COURSES OF STUDY IN MATHEMATICS FOR SENIOR HIGH SCHOOLS

Plane Geometry
Algebra III and IV
Solid Geometry
Plane Trigonometry

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FOREWORD

The purpose of this course of study in Senior High School Mathematics is to aid teachers in organizing the work for these grades. The material presented in the course has been tried out in connection with the work of a number of teachers and has been found most helpful.

Mathematics has a real contribution to make in the field of education. Boys and girls need to learn to use mathematics both in relation to courses in engineering, business, and in other fields, and to secure a clear conception of the important place which numbers and mathematical principles play in our everyday life. Many people think of mathematics as merely a collection of examples and problems. One who knows the theory underlying mathematics is in a position not only to use mathematical principles, but also to appreciate the uses to which mathematics is put in all forms of present day life. A well trained individual uses mathematics exactly as he would use other instruments and devices both in his professional and in his social and recreational life.

The materials presented in these outlines are considered basic to the mastery of regular courses in Senior High School Mathematics. Courses must be adapted to local groups and local conditions. Much experimentation is needed to determine best methods, the relationship and use of mathematics in science and other fields, and in the determination of courses in Senior High School Mathematics of a more general nature for those who do not desire to continue mathematics for more than a portion of their senior high school career. In an effort to solve these problems, we enlist the interest and help of the teachers of the Commonwealth who use this course.

This course is a part of a general program of curriculum revision organized under the direction of William H. Bristow, Chief, Education Bureau. The general committee on mathematics consists of:

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May 1, 1933

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CONTENTS

Page

FOREWORD 3

INTRODUCTION
- Trends in Instruction 7
- Plane Geometry 8
- Algebra III and IV 9
- Solid Geometry 10
- Plane Geometry 10

PLANE GEOMETRY
- Guiding Principles 11
- Introductory Unit 11
- Assumptions and Theorems 13
- Fundamental Theorems 14
- Fundamental Constructions 16
- Subsidiary Theorems 16
- An Arrangement of Content 18
- Unit Tests 20
- General Test 24

ALGEBRA III
- Unit I —Review of Fundamental Operations 26
- Unit II —Factoring 26
- Unit III —Review of Fractions 28
- Unit IV —Powers (Involution) 29
- Unit V —Roots (Evolution) 31
- Unit VI —The Theory of Exponents 32
- Unit VII —Radicals 34
- Unit VIII—Imaginary Numbers 35
- Unit IX —Quadratic Equations in One Unknown 36
- Unit X —Equations in Quadratic Form (Optional) 37
  - Unit Tests 48

ALGEBRA IV
- Unit I —Equations in Quadratic Form 38
- Unit II —Properties of Quadratics 38
- Unit III —Maxima and Minima 39
- Unit IV —Systems of Quadratics 40
- Unit V.. —Ratio, Proportion, and Variation 41
- Unit VI —Logarithms 42
- Unit VII —Arithmetic and Geometric Progressions 44
- Unit VIII—Binominal Theorem 45
- Unit IX —Numerical Trigonometry of the Right Triangle 46
- Additional Units 48
  - Unit Tests 57

Page

SOLID GEOMETRY

Guiding Principles 65
Introductory Unit 65
Fundamental Theorems 66
Subsidiary Theorems 68
Six Teaching Units 69
Unit Tests 70
General Test 71

PLANE TRIGONOMETRY

Unit I —Trigonometric Functions of Acute Angles .. 73
Unit II —Solution of Triangles 75
Unit III —Trigonometric Functions of Any Angle 76
Unit IV —Fundamental Identities 79
Unit V —Logarithms 79
Unit VI —Solution of Triangles by Use of Logarithms .. 80
Unit VII —Supplementary Topics 81
Unit Tests 81

COURSES OF STUDY IN MATHEMATICS FOR SENIOR HIGH SCHOOLS

INTRODUCTION

CONSTRUCTION of courses of study involves the determination of what purpose they are to serve. The purpose conditions the entire content. The elementary course of study in arithmetic is developed from the consumer's point of view, that is, the materials of the course are organized on a basis of the immediate needs and desires of the pupils. Life situations which are and will be of value to the child are stressed. Skill in the manipulation of numbers, growth in mathematical concepts, and the development of a sense of the value of the subject naturally grow out of this.

The junior high school course in mathematics is a continuation of the elementary work stressing individualization so that each child may increase his ability to apply the number concepts to his social world. The choice of problems is determined by and related to the child's needs. These problems are not considered as drill materials but as real situations. For example, problems in insurance are not taught with the idea of increasing speed in the fundamental operation or developing skill to compute premiums. They are presented so that comparison is made between the returns from this investment and those of other investments. Considerable integration of arithmetic, algebra and plane geometry has been accomplished in this course.

Noteworthy progress has been made in recent years in the teaching of mathematics in high school. The trends before 1923 were well summarized in the report of the National Committee (The Reorganization of Mathematics in Secondary Education, 1923). Later developments may be found in recent books and mathematical journals (The Mathematics Teacher, National Council of Teachers of Mathematics, Lancaster, Pennsylvania, and School Science and Mathematics, 2059 East 72nd Place, Chicago, Illinois), and in educational journals of a more general nature.

This course of study for the senior high school has been developed with the idea of filling the needs of two groups of pupils; those who have ability and interest but have little thought of its future use; and those who desire it as the basis for further work in science and mathematics. In preparing such a course the scope of materials to be included had first to be determined. The fundamental principles to be included were determined thru a careful study of the purposes of pupils who study these subjects. An effort has been made to relate them as closely as possible to those normal activities of the social and business world. These correlate very closely with the important recommendation of the National Committee on the Reorganization of Mathematics in Secondary Education. Pupils should become familiar with the fundamental principles outlined in the course. The emphasis to be placed on the various principles is left to the judgment of the individual teacher.

The principles included are organized into definite teaching units. A tentative outline of units with the estimated time for teaching each unit has been prepared. Actual experience of the teachers of the subjects was utilized in formulating suggested teaching procedures. The entire course has been tried out in a number of schools of the State. With this experience desirable changes were made. These are now offered as methods that have worked well in the actual teaching field. Objective tests and historical materials are included for those teachers who are teaching the course for the first time. The teacher should work out tests for his prospective class. The use of aids such as rulers, compasses, protractors, coordinate paper, geometric figures, drawing boards, T square, pantograph, mimeographing machine and supplies, and slide rule will aid much in vitalizing instruction.

The general objectives for the mathematics course of study for the senior high school are:

1. Continued application of the skills learned in the preceding mathematics courses.
2. Accuracy and facility in the fundamental processes.
3. Knowledge and power to apply mathematic concepts to such problems as are encountered in physics, chemistry, statistical materials and other fields.
4. Specific knowledge of mathematics useful in interpreting economic and social problems.
5. Development of logical reasoning as it relates to mathematics.
6. General cultural influence, through ability to interpret life problems.
7. Stimulation of intellectual interests for those with ability in this field.

PLANE GEOMETRY

In view of the fact that most pupils are interested in certain phases of plane geometry, a number of adjustments have been made. Included among these are: a careful introduction, the elimination of many theorems which are now generally considered of traditional value (see list of theorems of the National Committee, the College Entrance Examination Board and the New York State Syllabus; observe that there are no corollaries in these lists), and emphasis on functioning thinking, in terms of the relation between quantities. The following important tendencies are also given careful consideration:

1. A lessening of the vocabulary load. (Terms such as isosceles and mutually equilateral are not used in the theorems of Plane Geometry in the lists of the National Committee, the College Entrance Examination Board, or the New York State Syllabus. See also pp. 74-77 of the National Committee Report and the present course of study, Page 11, No. 8. Studies of vocabularies for various high

school subjects are published in the Journal of Educational Research.)

2. Care in matters of definition. (For example, "axiom" is no longer defined as a "self-evident truth," but is considered as an assumption.) Memorizing a formal definition is not sufficient. The pupil must have many and varied experiences in the use of a word before he really comprehends it.

3. Connections of geometry with life. The subject becomes more vital, especially to the beginner, if the manifold uses of geometry are stressed. For example, let the pupil find straight lines, angles, triangles, circles, etc.:

    a. in nature—birds' nests, edges of leaves, flowers, frost patterns, honeycombs, phases of the moon, rainbows, sea urchins, seed pods, snow flakes, spider webs, starfishes;

    b. in buildings, carriers, and machinery—airplanes, automobiles, a belt over two or more pulleys, bridges, Eskimo igloos, Grecian and Roman buildings, Indian wigwams, modernistic buildings, ocean liners;

    c. in maps and plans of buildings, machines, parks and cities;

    d. in designs on carpets, dishes, dresses, jewelry, oil cloth, tiles, wall paper, windows;

    e. in athletic fields—baseball diamond, basketball court, and tennis court.

Algebra III and IV

In administering the courses in algebra III and IV, it is urged that the functional aspect of the equation be constantly stressed. Leading authorities[1] on the teaching of algebra show the importance of such a presentation. Every equation is a statement of the relation existing between two or more unknowns. Therefore, it is only logical that students be made conscious of this explicit or implied interdependence between the unknowns. Whatever other mathematical concepts of the equation are taught this functional concept should not be overlooked. To overlook it makes our teaching of the equation incomplete and inadequate. This concept glorifies the usefulness and meaning of the equation. It cannot be overstressed. In many instances it has been overlooked entirely in our teaching.

To increase one's aptness in teaching such topics as the binomial theorem; graphs; the theory of exponents, teachers are urged to read pp. 24-34 of the Third Yearbook.[2] The applications of these mathematical theories to everyday life vitalize the teaching of modern mathematics. How much more interesting could the topic on radicals be made if in addition to knowing the value of the square roots of two and three, we show our pupils the practical use that artists make of these incommensurable ratios in the matter of dynamic symmetry.[3]

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[1] The National Council of Teachers of Mathematics, The Third Yearbook pp. 42-56, Bureau of Publications, Teachers College, Columbia University, 525 West 120th Street, New York City.

[2] Ibid 24-34

It is recommended that every teacher of mathematics study the recommendations made in the Third Yearbook.[4]

The courses have been planned to present to the students those theories and processes in the subject regardless of the fact that he may or may not intend to pursue further his study in the field of mathematics. The courses comprise ten units in each division. Should these units be taught in less than the allocated time the additional time should be used in a general review of the entire course or in a review of such units in which the pupils have shown a deficiency of attainment. The teacher should aim at thoroughness rather than at the idea of attempting units in a superficial way.

Solid Geometry

In Solid Geometry, as in Plane Geometry, there is a strong tendency to reduce the number of theorems and to lessen the vocabulary load.

By the assumption (1) of the formula for the volume of a rectangular solid (a rectangular parallelepiped), (2) of Cavalieri's theorem, and (3) of the prismatoid formula it is possible to simplify and to generalize the theorems in mensuration. The time thus gained can be profitably devoted to abundant practice in numerical problems and to the development of accurate space perceptions.

The question of the possibility and the desirability of replacing the usual one year course in Plane Geometry by a one year course including parts of both Plane and Solid Geometry has been discussed at length. But it will be necessary to make many more experiments and to secure many more facts before it is possible to reach any valid conclusion in this matter.

Plane Trigonometry

This outline for a one term course reflects in its content the modern point of view. An effort should be made on the part of the teacher to vitalize the subject by introducing practical and applied problems from related fields. The course has been developed logically through the use of angle functions as the unifying thread. It is believed that this course should offer a genuine opportunity for real mathematical thinking by placing emphasis on this aspect of the work, rather than on the mere acquisition of certain skills. Visualization and logical reasoning rather than memory work should be emphasized throughout.

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[3] Ibid 57-64

[4] Ibid

PLANE GEOMETRY

GUIDING PRINCIPLES

1. Proceed slowly at first, making sure that the student understands each topic before beginning the next. Show the student how to study.

2. Give simple exercises immediately after definitions, assumptions, constructions, and theorems. Include constructions and numerical work in the graded exercises.

3. Be sure that all assignments are definite.

4. Make provision for differences in ability. (For example, give additional exercises or topics for the more able students.)

5. Plan for every lesson, no matter how many times you have taught the subject. But be ready to modify your plan to meet an emergency.

6. Vary your methods sufficiently to avoid monotony. Adapt your methods to the needs of your students.

7. Make as many contacts as possible with other subjects; as, building houses, bridges, and airplanes (rigidity of the triangle); shop work (testing a core box, finding center of circular rod); art and architecture (circles, triangles, and rectangles in decoration).

8. Do not attempt to give formal definitions of terms such as point, straight line, plane, and magnitude, because there are no simpler ideas in which to phrase the definitions. Definitions should be introduced as they are needed. Memorizing a formal definition is not sufficient; the student must grasp the meaning of the term and know how to use it. ''Circle'' is the curve; ''circumference'' is its length; its ''area'' is the numerical measure of the portion of the plane inclosed by the circle. ''Polygon,'' its ''perimeter,'' and its ''area'' are defined in like manner. An ''obtuse angle'' is greater than 90° and less than 180°. Some terms can be profitably abandoned; such as, antecedent, consequent, conjugate angles, equivalent, homologous, perigon, reflex angle, rhomboid, scalene, trapezium, Q. E. D. and Q. E. F. Avoid the introduction of unnecessary new terms.

INTRODUCTORY UNIT—SUGGESTED SEQUENCE

1. Relation of geometry to everyday life

   Circles, squares, triangles, cones, pyramids, and other geometric figures found in nature, art, and industry (See Foreword)

2. Value of geometry

3. Study suggestions

4. Discussion of familiar terms

   Straight line, point, etc.

5. Definitions

   Line segment, straight angle, right angle, acute angle, obtuse angle

6. The use of algebra in geometry

   Review of axioms on equals used by student in algebra

   Simple equations in which the unknown quantities are angles or line segments

7. Discussion*

   Assumptions 1-5

   Constructions 1 and 2

8. Definitions

   Adjacent angles, complementary angles, supplementary angles, vertical angles

9. Discussion

   Assumptions 6-9

10. Definition

    Perpendicular lines

11. Discussion

    Construction 3

    Assumptions 10 and 11

12. Definitions

    Perpendicular bisector of a line segment, bisector of an angle

13. Discussion

    Constructions 4 and 5

14. Definition

    Parallel lines

15. Discussion

    Construction 6

    Assumptions 12-14

16. Meaning of terms

    Rectangle

    Area

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*For Assumptions and Constructions referred to in the odd numbered steps 7 to 19 inclusive, see Pages 13, 14 and 16.

17. Discussion

    Assumption 15

    (This can be deferred to Unit XVI if desired)

18. Explanation of terms

    Congruent triangles

    Corresponding sides (angles) of congruent triangles

    (Stress the fact that corresponding sides (angles) of congruent triangles are equal)

19. Discussion

    Assumption 16

20. Discussion of Fundamental Theorem I

    Use actual superposition of wire models or of triangular pieces of paper. Theoretical superposition can be used later, if desired.

    Emphasize the rigidity of the triangle

21. Formal proof of one or more of the simple theorems already discussed. (For example, vertical angles are equal)

    Advantages of formal proof

It is suggested that six weeks be given to this unit

If the student has had a course in intuitive geometry, less time will be needed

ASSUMPTIONS AND THEOREMS FOR INFORMAL TREATMENT

1. Through two distinct points it is possible to draw one straight line, and only one.

2. A line segment can be produced to any desired length.

3. The shortest path between two points is the line segment joining them.

4. From a given center and with a given radius one and only one circle can be described in a plane.

5. A straight line intersects a circle in at most two points.

6. All right angles are equal.

7. If the sum of two adjacent angles equals a straight angle, their exterior sides form a straight line.

8. Equal angles have equal complements and equal supplements.

9. Vertical angles are equal.

10. One and only one perpendicular can be drawn through a given point to a given straight line.

11. The shortest distance from a point to a line is the perpendicular distance from the point to the line.

12. Through a given point not on a given straight line, one straight line, and only one, can be drawn parallel to the given line.

13. Two lines parallel to the same line are parallel to each other.

14. Two lines perpendicular to the same line are parallel.

15. The area of a rectangle is equal to its base times its altitude.

16. Any figure may be moved from one place to another without changing its shape or size.

FUNDAMENTAL THEOREMS

1. Two triangles are congruent if (a) two sides and the included angle of one are equal respectively to two sides and the included angle of the other; (b) two angles and the included side of one are equal respectively to two angles and the included side of the other; (c) the three sides of one are equal respectively to the three sides of the other.

2. If two sides of a triangle are equal, the angles opposite these sides are equal; and conversely.

3. Two right triangles are congruent if the hypotenuse and one other side of one are equal respectively to the hypotenuse and one other side of the other.

4. When a transversal cuts two parallel lines, the alternate interior angles are equal; and conversely.

5. The sum of the angles of a triangle is two right angles.

6. A parallelogram is divided into congruent triangles by either diagonal.

7. Any quadrilateral is a parallelogram (a) if the opposite sides are equal; (b) if two sides are equal and parallel.

8. If a series of parallel lines cut off equal segments on one transversal, they cut off equal segments on any transversal.

9. The locus of a point equidistant from two given points is the perpendicular bisector of the line segment joining them.

10. The locus of a point equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by these lines.

11. In the same circle or in equal circles, if two arcs are equal, their central angles are equal; and conversely.

12. In any circle, angles at the center are proportional to their intercepted arcs. (Proof for commensurable case only.)

13. In the same circle or in equal circles, if two arcs are equal their chords are equal; and conversely.

14. (a) A diameter perpendicular to a chord bisects the chord and the arcs of the chord. (b) A diameter which bisects a chord (that is not a diameter) is perpendicular to it.

15. In the same circle or in equal circles, equal chords are equally distant from the center; and conversely.

16. The tangent to a circle at a given point is perpendicular to the radius at that point; and conversely.

17. An angle inscribed in a circle is equal to half the central angle having the same arc.

18. Angles inscribed in the same segment are equal.

19. (a) If a straight line is drawn through two sides of a triangle parallel to the third side, it divides these sides proportionally. (b) If a line divides two sides of a triangle proportionally, it is parallel to the third side. (Proof for commensurable case only.) (c) The segments cut off on two transversals by a series of parallels are proportional.

20. Two triangles are similar if (a) they have two angles of one equal respectively to two angles of the other; (b) they have an angle of one equal to an angle of the other and the including sides are proportional; (c) their sides are respectively proportional.

21. If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.

22. The bisector of an (interior or exterior) angle of a triangle divides the opposite side (produced if necessary) into segments proportional to the adjacent side.

23. In any right triangle the perpendicular from the vertex of the right angle to the hypotenuse divides the triangles into two triangles similar to the given triangle.

24. In a right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

25. The perimeters of two similar polygons have the same ratio as any two corresponding sides.

26. Polygons are similar, if they can be decomposed into the same number of triangles, similar each to each and similarly placed; and conversely.

27. (a) The area of a parallelogram is equal to the base times the altitude. (b) The area of a triangle is equal to one-half the base times the altitude. (c) The area of a trapezoid is equal to half the sum of its bases times its altitude.

28. The areas of two similar triangles (or polygons) are to each other as the squares of any two corresponding sides.

29. The area of a regular polygon is equal to half the product of its apothem and perimeter.

30. If a circle is divided into equal arcs, the chords of these arcs form a regular inscribed polygon, and tangents at the points of division form a regular circumscribed polygon.

31. The circumference of a circle is equal to $2 \pi r$ (informal proof only).

32. The area of a circle is equal to πr^2 (informal proof only).

FUNDAMENTAL CONSTRUCTIONS

1. Construct an angle equal to a given angle.

2. Construct a triangle, given (a) the three sides; (b) two sides and the included angle; (c) two angles and the included side.

3. Construct a perpendicular to a given line through a given point.

4. Bisect a line segment and draw the perpendicular bisector.

5. Bisect an angle.

6. Through a given point construct a straight line parallel to a given straight line.

7. Given an arc of a circle, find its center.

8. Circumscribe a circle about a triangle.

9. Inscribe a circle in a triangle.

10. Construct a tangent to a circle through a given point.

11. Divide a line segment into parts proportional to given segments.

12. Construct the fourth proportional to three given line segments.

13. Construct the mean proportional between two given line segments.

14. Construct a triangle (polygon) similar to a given triangle (polygon).

15. Construct a triangle equal in area to a given polygon.

16. Inscribe a square in a circle.

17. Inscribe a regular hexagon in a circle.

SUBSIDIARY THEOREMS

1. When two lines are cut by a transversal, if the corresponding angles are equal or if the interior angles on the same side of the transversal are supplementary, the lines are parallel.

2. When a transversal cuts two parallel lines, the corresponding angles are equal, and the interior angles on the same side of the transversal are supplementary.

3. A line perpendicular to one of two parallels is perpendicular to the other also.

4. If two angles have their sides respectively parallel or respectively perpendicular to each other, they are either equal or supplementary.

5. Any exterior angle of a triangle is equal to the sum of the two opposite interior angles.

6. The sum of the angles of a convex polygon of n sides is $2(n-2)$ right angles.

7. In any parallelogram (a) the opposite sides are equal; (b) the opposite angles are equal; (c) the diagonals bisect each other.

8. Any quadrilateral is a parallelogram, if (a) the opposite angles are equal; (b) the diagonals bisect each other.

9. The bisectors of the angles of a triangle meet in a point which is equidistant from the sides.

10. The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the vertices.

11. The altitudes of a triangle meet in a point.

12. The medians of a triangle meet in a point which is two-thirds of the distance from a vertex to the midpoint of the opposite side.

13. (a) If two sides of a triangle are unequal, the greater side has the greater angle opposite it; and conversely. (b) If two sides of one triangle are equal respectively to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third of the first is greater than the third side of the second; and conversely.

14. The tangents to a circle from an external point are equal.

15. Parallel lines tangent to or cutting a circle intercept equal arcs on the circle.

16. An angle inscribed in a semicircle is a right angle.

17. An angle formed by a tangent of a circle and a chord drawn through the point of contact is measured by half the intercepted arc.

18. An angle formed by two intersecting chords is measured by half the sum of the intercepted arcs.

19. An angle formed by two secants or by two tangents to a circle is measured by half the difference between the intercepted arcs.

20. (a) In the same circle or in equal circles if two chords are unequal, the greater is at the less distance from the center; and conversely. (b) In the same circle or in equal circles the greater of two minor arcs has the greater chord, and conversely.

21. If from a point without a circle a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.

22. Parallelograms or triangles of equal bases and equal altitudes are equal in area.

23. The perimeters of two regular polygons of the same number of sides are to each other as their radii and also as their apothems.

AN ARRANGEMENT OF THE CONTENT IN EIGHTEEN TEACHING UNITS

Unit One. Introduction (Suggested time—Six weeks)

The sixteen Assumptions, Constructions 1-6, and Fundamental Theorem 1.

Unit Two. Triangles (One week)

Fundamental Theorems 2 and 3. Some teachers defer the proof of the converse in Theorem 2.

Unit Three. Parallel Lines and Perpendicular Lines (One week)

Fundamental Theorem 4 and Subsidiary Theorems 1-4.

Unit Four. Sums of Angles (One week)

Fundamental Theorem 5 and Subsidiary Theorems 5 and 6.

Before Subsidiary Theorem 6 is stated, let the pupil find the sum of the angles of polygons of four, five, six, or more sides by dividing them into triangles.

Unit Five. Parallelograms (One week)

Fundamental Theorems 6-8 and Subsidiary Theorems 7 and 8.

Review one week.

Unit Six. Locus (One week)

Fundamental Theorems 9 and 10. Before studying these theorems give many simple exercises in discovering a locus by locating many points that obey the given law. Do not attempt to give formal proofs for these exercises.

Unit Seven. Lines Through a Point (One week)

Subsidiary Theorems 9-12. Let the pupil locate these points in an equilateral triangle, a right triangle, an obtuse triangle, etc.

Unit Eight. Inequalities (in triangles) (One week)

Subsidiary Theorem 13. This unit may be deferred until Unit Eleven has been completed.

Review one week.

Unit Nine. Arcs, Chords, and Central Angles (Two weeks)

Fundamental Theorems 11-15 and Constructions 7 and 8.

Unit Ten. Tangents and Secants (One week)

Fundamental Theorem 16, Subsidiary Theorems 14 and 15, and Constructions 9 and 10.

Unit Eleven. Measurement of Angles (Two weeks)

Fundamental Theorems 17 and 18 and Subsidiary Theorems 16-19.

Be sure that the student knows which arcs are intercepted by these angles.

Unit Twelve. Inequalities (in Circles) (One week)

Subsidiary Theorem 20. The theorems on inequalities (in triangles and in circles) afford an opportunity for a simple discussion of functional relationships.

Review one week.

Unit Thirteen. Proportions (One week)

A proportion may be treated as a fractional equation. The transformations may be obtained by algebraic processes.

Unit Fourteen. Proportional Line Segments and Similar Triangles (Five weeks)

Fundamental Theorems 19-24, Subsidiary Theorem 21, and Constructions 11-13. Develop the concepts of corresponding angles and corresponding sides of similar triangles before proving theorems on similar triangles.

Unit Fifteen. Similar Polygons of n Sides (One week)

Fundamental Theorems 25 and 26 and Construction 14.

Let this work be closely connected with the work of the last unit.

Review one week.

Unit Sixteen. Areas (Two weeks)

Fundamental Theorems 27 and 28, Subsidiary Theorem 22, and Construction 15. Fundamental Theorem 28 needs careful attention.

Unit Seventeen. Circles and Regular Polygons (One week)

Fundamental Theorems 29 and 30, Subsidiary Theorem 23, and Constructions 16 and 17.

Unit Eighteen. Measurement of the Circle (One week)

Fundamental Theorems 31 and 32. Connect this unit closely with the last unit. Give informal work only.

Review two weeks.

The time suggested for these eighteen units is thirty-six weeks. If the school year is longer than nine months, there will be opportunity to spend more time on some of these units, or to study additional topics such as mean and extreme ratio, the sine, cosine, and tangent of an acute angle, or the areas of pyramids and prisms.

UNIT TESTS—PLANE GEOMETRY

The following tests are offered as suggestions for simple half-hour tests. In all completion tests three dots indicate that one or more words have been omitted.

Test on Unit One

1. (a) Construct a triangle when the three sides are given. (b) Draw a line segment. Construct its perpendicular bisector.

2. (a) There are ..., ..., and ... degrees in the three other angles formed by two intersecting lines if one of the angles is 75° 15′. (b) The hour hand of a clock moves through ... degrees in two hours.

3. (a) The supplement of ... angle is an obtuse angle. (b) The supplement of ... angle is a right angle. (c) The complement of an angle of 35° 24′ equals ... degrees.

4. (a) In triangles PQR and DEF if side PQ equals side EF, side RQ equals side DF, and angle ... equals angle ..., then the two triangles are congruent. (b) In triangles MNO and XYZ if angle M equals angle X, angle O equals angle Y, and side ... equals side ..., then the two triangles are congruent.

Test on Unit Two

1. If two sides of a triangle are equal, the ... these sides are .... Prove. Write the converse of this theorem. Is it true?

2. (a) In triangle ABC if side AB equals 1½ in., side CB equals 1½ in., and angle C equals 45°, angle A equals ... Why? (b) In triangle FGH if angle F equals 30° and angle H equals 30°, side ... equals side ...

3. If angle P equals 30°, angle Q equals 60°, and angle R equals 90°, triangle PQR is a ... triangle because ..., and side PQ is called the ... because ... Two right triangles are congruent if ...

4. In triangle ABC if AD is perpendicular to BC and CE is perpendicular to AB and AD equals CE, why are triangles ADC and AEC congruent? Is side AB equal to side CB? Why?

Test on Unit Three

1. Prove:—If two parallel lines are cut by a transversal, the alternate interior angles are equal. State the converse. Is it true?

2. If two lines are cut by a transversal so that one of two corresponding angles equals 45° and the other is one-third of the supplement of 45°, are the lines parallel? Why?

3. If two parallel lines are cut by a transversal so that one of the eight angles formed equals 75°, how many degrees are there in each of the seven other angles?

4. (a) If a line is perpendicular to one of two parallel lines, is it parallel or perpendicular to the other? Why? (b) If two lines are

perpendicular to the same line, are they perpendicular or parallel to each other? Why?

Test on Unit Four

1. Prove:—The sum of the angles of a triangle is two right angles.

2. In triangle ABC if angle ABC equals 45° and exterior angle BAX equals 120°, angle ACB equals . . . degrees. Why?

3. (a) How many degrees are there in each angle of a regular polygon of five sides? of twelve sides? (b) How many sides has a regular polygon if one of its angles equals 144°? One and one-third right angles?

4. In quadrilateral ABCD if angle B equals twice angle A, angle C equals three times angle A, and angle D equals four times angle A, how many degrees are there in each of these four angles?

Test on Unit Five

1. Prove:—Any quadrilateral is a parallelogram if two sides are equal and parallel.

2. If one angle of a parallelogram is a right angle, why are the three other angles right angles?

3. Are the diagonals of a rectangle equal? Why? Are they perpendicular to each other? Why? Do they bisect each other? Why?

4. In parallelogram ABCD if angle A equals 60°, is AC equal to BD? Why? Is angle A equal to angle C? Why? Are angles A and B complementary or supplementary? Why? Does AC bisect angle A? Why?

Test on Unit Six

1. Draw a straight line and call it XY. Locate a point which is 1 in. from XY. Locate nine other points each 1 in. from XY. Are these ten points on a circle, or one line, or two parallel lines, or two intersecting lines?

2. What is the locus of the point of a round lead pencil which rolls along the top of a desk or table?

3. Prove:—The locus of a point equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by these lines.

Test on Unit Seven

1. Explain a method of locating a point which is equidistant from the sides of a triangle.

2. Prove:—The perpendicular bisectors of the sides of a triangle meet in a point which is equidistant from the vertices.

3. What is a median of a triangle? In triangle ABC medians AD and BE intersect at O. If AD equals 1½ in. and BE equals 2¼ in., find AO, BO, DO, and EO.

Test on Unit Eight

1. Prove:—If two sides of a triangle are unequal, the greater side has the greater angle opposite it.

2. Why is the hypotenuse of a right triangle the longest side? Is it possible to draw triangle ABC with AB the longest side and angle C an acute angle? Why?

3. In triangle PQR side QR is greater than side PR and angle P equals 35°. Is angle R acute, right, or obtuse? Why?

4. In parallelogram ABCD the diagonals intersect at O and angle AOB is obtuse. Is AB greater than AD, or equal to AD, or less than AD? Why?

Test on Unit Nine

1. Prove:—A diameter which bisects a chord (that is not a diameter) is perpendicular to it.

2. A circle passes through the vertices of triangle ABC. If arcs AB, BC, and CA are equal, why is triangle ABC equilateral?

3. Point O is the center of a circle. If arc AB equals 60° and arc CD equals 120°, is angle COD equal to twice angle AOB? Why? Is chord CD equal to twice chord AB? Why?

4. Explain how to find the center of a circle when an arc is given.

Test on Unit Ten

1. Prove that the tangent to a circle at a given point is perpendicular to the radius at that point.

2. If AB is a diameter of a circle, why are the perpendiculars to AB at points A and B tangent to the circle?

3. Why is a trapezoid inscribed in a circle an isosceles trapezoid?

4. If quadrilateral ABCD is circumscribed about a circle, why is the sum of sides AB and DC equal to the sum of sides BC and AD?

Test on Unit Eleven

1. Draw a circle and locate points A, B, C, and D on it so that arc AB equals 60°, arc BC equals 45°, and arc CD equals 120°. Use this circle for questions 2, 3, and 4.

2. Draw chords BA, BC and BD. Angle ABC equals ... degrees; angle ABD equals ... degrees. Is chord BD a diameter? Why?

3. Construct the tangents at B and at C. Call their point of intersection E. Angle BEC equals...degrees; angle ECB equals...degrees.

4. Draw chord AC. It intersects BD at F. Is chord AC a diameter? Why? Angle AFB equals ... degrees.

Test on Unit Twelve

1. Prove:—If two chords of a circle are unequal, the greater is at the less distance from the center.

2. Are the bases of an inscribed trapezoid equally distant from the center of the circle? Why?

3. A square (ABCD) and a triangle (EFG) are inscribed in a circle. If triangle EFG is equilateral, chord AB is . . . the center than chord EF. Why?

4. Triangle ABC is inscribed in a circle. If arc AB equals 90° and arc BC equals 120°, which is the largest angle of the triangle? Why? Which is the shortest side? Why?

Test on Unit Thirteen

1. In triangle ABC point D is on side AB and point E is on side CB:—

a. if $\frac{AD}{DB} = \frac{CE}{EB}$, explain how $\frac{AB}{DB} = \frac{CB}{EB}$ is obtained.

b. if $\frac{AB}{DB} = \frac{CB}{EB}$, explain how $\frac{AD}{DB} = \frac{CE}{EB}$ is obtained.

e. if $\frac{AD}{DB} = \frac{CE}{EB}$, explain how $\frac{DB}{AD} = \frac{EB}{CE}$ is obtained.

d. if $\frac{AD}{DB} = \frac{CE}{EB}$, explain how $\frac{AD}{CE} = \frac{DB}{EB}$ is obtained.

Test on Unit Fourteen

1. Prove:—Two triangles are similar if their sides are respectively proportional.

2. (a) Are triangles ABC and A′B′C′ similar if angle A equals 65°, angle A′ equals 65, angle B equals 45°, and angle B′ equals 45°? Why? (b) Are triangles DEF and GHK similar if angle E equals 35°, angle H equals 35°, and sides DE, EF, GH, and HK are equal respectively to 1 in., 1½ in., 1½in., and 2¼ in.? Why? (e) Are triangles JKM and NPQ similar if sides JK, KM, MJ, NP, PQ, and QN are equal respectively to 1½ in., 1 in., 2 in., 1 in., ½ in., and ¾ in.? Why?

3. (a) In triangle XYZ if angle Y equals 90°, side XY equals 1½ in., side ZY equals 2 in., then side XZ equals . . . in. (b) The center of a circle is O and its radius equals 2½ in. If line segment OP equals 6½ in., the tangent from P equals . . . in.

Test on Unit Fifteen

1. The sides of a polygon are equal to 1 in., 1½ in., 1¾ in., 2 in., and 1¾ in. Find the perimeter of a similar polygon whose shortest side is 4 in.

2. The sides of a polygon are 2 in., 3 in., 4 in., 5 in., and 6 in. Find the sides of a similar polygon whose perimeter is 35 in.

3. Draw a polygon (ABCDE). Draw a line segment A′B′ which is less than AB. Construct polygon A′B′C′D′E′ similar to ABCDE.

4. Two polygons are similar if they can be ... The converse is ...

Test on Unit Sixteen

1. Prove that the area of a trapezoid is equal to one-half its altitude times the sum of its bases.

2. The perimeter of an equilateral triangle is 36 in. Its altitude is ...

3. A cement walk 5 ft. wide is laid around the outside of a lot 20 ft. by 110 ft. The area of the walk is ...

4. If the ratio of the bases of two rectangles is two-thirds and the ratio of their altitudes is one-half, the ratio of their areas is ...

Test on Unit Seventeen

1. Prove:—The area of a regular polygon is equal to one-half the product of its apothem and its perimeter.

2. Draw a circle. Inscribe a regular polygon of twelve sides.

3. Draw a circle. Circumscribe a regular polygon of eight sides.

Test on Unit Eighteen

1. The radius of a circle is 7 in. Its circumference is ... Its area is ...

2. How many revolutions does a wheel 28 in. in diameter make in going one mile?

3. Find the number of square feet in the area of a walk 2 yds. wide surrounding a circular pond whose diameter is 100 ft.

4. Find the circumference of a circle inscribed in an equilateral triangle whose side equals two inches.

A GENERAL TEST ON PLANE GEOMETRY

This examination consists of three parts, which the teacher may find it convenient to give on three different days. Parts I and II are based on Fundamental Theorems and Constructions; Part III involves the Subsidiary Theorems also. If a student takes all three parts, Part I receives 30%, Part II 40%, and Part III 30%. The student who has not studied the Subsidiary Theorems will not take Part II and will receive 50% on Part I and 50% on Part II.

Part I

Answer these questions, but do not write out proofs.

1. Find the area of a circle whose diameter is 14 inches.

2. Find the area of triangle ABC if side AB = 9 inches, side BC = 6 inches, and angle B = 90°.

3. Find the altitude of a trapezoid whose area is 52 square inches and whose bases are 10 inches and 16 inches.

4. What is the length of a diagonal of a square whose side equals 9 inches?

5. Find the area of parallelogram ABCD if AB = 8 inches, AD = 6 inches, and angle A = 60 degrees.

6. If the radius of a circle equals 3 inches, how many inches are there in the arc intercepted by an inscribed angle of 30°?

7. What is the locus of points 1 inch from a given straight line?

8. If the radii of two circles are 2 inches and 3 inches, what is the ratio of the area of the first circle to the area of the second?

9. If the sides of a polygon are 1¾ in., 1¾in., 1½ in., 1¾ in., and 2¼ in., how many inches are there in the perimeter of a similar polygon whose shortest side is 1 in.?

10. The area of a polygon is 36 sq. in. and one side is 2 in. Find the area of a similar polygon whose corresponding side is 3 in.

Part II (Proofs)

1. Prove:—An angle inscribed in a circle is equal to half the central angle having the same arc.

2. Prove:—A diameter perpendicular to a chord bisects the chord and the arcs of the chord.

3. Prove:—Two triangles are similar if an angle of one equals an angle of the other and the including sides are proportional.

4. Prove:—The locus of a point equidistant from two given points is the perpendicular bisector of the line segment joining them.

Part III

In answering these questions, do not write out a proof unless you are asked to do so.

1. The altitude of an equilateral triangle is 4½ in. Find the radius of the inscribed circle and the radius of the circumscribed circle.

2. How many sides has a regular polygon if an angle equals 165°?

3. Prove:—If two chords of a circle are unequal, the greater is at the less distance from the center.

4. Chords AC and BD of a circle intersect at P and secants AD and BC of that circle intersect at R. Find the number of degrees in angle DPC and in angle DRC if the intercepted arcs AB and CD equal 60° and 120° respectively.

5. If angle ABC equals 30° and is formed by tangent AB and chord BC, how many degrees are there in inscribed angle BDC whose vertex is on minor arc BC?

6. Point O is the center of a circle whose radius is 3 in. If line segment PO equals 5 in., what is the length of a tangent to the circle from P?

ALGEBRA III

Unit One. Review of Fundamental Operations (Seven days)

In schools where the course in algebra three follows immediately algebra one and two, this unit may be omitted. If, however, this unit is used, the following suggestions pertain.

A rapid review of the four fundamental operations of addition, subtraction, multiplication, and division, should be given. Stress should be laid upon the algebraic laws governing these operations. In addition to the above, the removal and insertion of parentheses, special methods of multiplying, and simple equations—integral, fractional, and literal—should be reviewed.

Problems stressing these fundamentals may be secured from the textbook in use or in practice pads in algebra.

Unit Two. Factoring (Eight days)

The unit in factoring should include a review of the types in factoring taught in the preliminary courses in algebra and, in addition, a study of such types included below:

TYPES I Polynomials

- a. $ax + ay + am + an$
- b. $ax + ay + bx + by$

II Trinomials

- a. $a^2 \pm 2ab + b^2$ Perfect square trinomial
- b. $ax^2 + bx + c$ General quadratic trinomial

III Binomials

- a. $4a^4 + b^4$, $a^6 + b^6$ or $a^2 + b^2$
- b. $a^2 - b^2$
- c. $a^5 + b^5$
- d. $a^5 - b^5$

IV The Factor Theorem

It will be seen that the types included above are sufficient to meet all demands in factoring that may be made upon the pupil subsequently.

SUGGESTIONS IN TEACHING THE UNIT

Recall to the class the definition of the term factor. Make it clear, by numerous examples, that when we are called upon to find the factors of an expression, we are to find *all* the *prime* numbers that multiplied will produce the given expression.

The following procedure should be mastered by the pupil:

TYPE I

A. Is there a common factor throughout the expression? If so, remove it by division. Use the highest factor.

Example: $ax + ay + am + an$

Solution: $a(x + y + m + n)$

B. If not, is there a common factor by grouping? If so, group and factor as in Type I, A, above.

Example: $ax + ay - bx - by$

Solution: $a(x + y) - b(x + y)$

$= (x + y)(a - b)$

TYPE II TRINOMIALS. There are two principal kinds of trinomials:

A. The perfect square trinomial. If it does not come under Type I, then is the expression a perfect square trinomial? That is, does it contain two positive squares and ± twice the product of the square roots of the squares? If so the factors are identical, being the square roots of the squares joined by the sign of twice the product.

Example: $4a^2 + 4ab + b^2 = (2a + b)(2a + b)$ or $(2a + b)^2$

Example: $4a^2 - 4ab + b^2 = (2a - b)(2a - b)$ or $(2a - b)^2$

B. If not, is the expression a general quadratic trinomial in the form $ax^2 + bx + c$? If so, factor by trial.

Example: $x^2 + 7x + 12 = (x + 3)(x + 4)$

Example: $4x^2 - 5x - 6 = (4x + 3)(x - 2)$

TYPE III BINOMIALS. There are four kinds:

A. The sum of the even powers, $a^2 + b^2$ which has no real factors, with the following exceptions:

1. $4a^4 + b^4$ may be factored by using Type II, A, above and then III, B, following.
2. $a^6 + b^6$ which may be written as the sum of two odd powers $(a^2)^3 + (b^2)^3$ and factored by type III, C, following.

B. The difference of two even powers, $a^2 - b^2$ has two factors, the product of the sum and difference of the square roots of the squares.

Example: $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2)$

$= (a^2 + b^2)(a + b)(a - b)$

C. The sum of the odd powers, $a^3 + b^3$, has two factors.

Example: $(a^5 + b^5) = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$

D. The difference of the odd powers, $a^3 - b^3$, has two factors.

Example: $(a^5 - b^5) = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$

TYPE IV THE FACTOR THEOREM

Example: $x^4 - x^3 - 7x^2 + x + 6$

Procedure: First arrange the terms in descending order.

Second, let x equal one factor of the last term, 6.

Let $x = 1$

Thirdly, detach the coefficients of the given problem supplying zero for any missing coefficients.

Fourthly, using the guessed factor, 1, as a divisor and applying Horner's method of synthetic division, determine whether the guessed factor is a true factor of the given expression.

By repeating this operation as often as is possible, all the factors of the given expression may be found.

The factors for the expression given above are

$(x - 1)(x + 1)(x - 3)(x + 2)$

Suggestion: After teaching each type as a unit, the teacher should insist that the pupil learn this scheme of factoring, so that he will be able to classify any problem in factoring under one of the four types listed above, and proceed to factor as suggested in that type.

Suggested topic for discussion: Horner's method of synthetic division.

Unit Three. Review of Fractions (Ten days)

I. SIGNS

Three signs are present, either expressed or implied, the sign of the numerator, the sign of the denominator, and the sign of the fraction: $-\dfrac{-a}{-b}$

A. Rules for changing signs when the terms of the fraction are not factored.

1. We may change all the signs of either the numerator, or the denominator, if we change the sign of the fraction.

$$\frac{a - b}{a - b - c} = -\frac{b - a}{a - b - c} = -\frac{a - b}{-a + b + c}$$

2. We may change all the signs of both the numerator and of the denominator without changing the sign of the fraction.

$$\frac{a - b}{a - b + c} = \frac{b - a}{- a + b - c}$$

B. Rules for changing signs when the terms of the fraction are factored.

1. We may change the signs of an odd number of factors if we change the signs of the fraction.

2. We may change the sign of an even number of factors without changing the sign of the fraction.

II. MULTIPLICATION

A. Find the prime factors of the numerator and denominator.

B. Divide by factors common to numerator and denominator.

C. Multiply the remaining factors of the numerator for the numerator of the product, and of the denominator for the denominator of the product.

NOTE: Students should remember that when all the factors in the numerator and denominator cancel the result is 1 not zero.

III. DIVISION

In division invert the divisor and proceed as in multiplication.

IV. ADDITION AND SUBTRACTION

A. Change the fractions to equivalent fractions having the lowest common denominator (L.C.D.).

B. Find the algebraic sum of the numerators and place the sum over the L.C.D.

C. Reduce the result to its lowest terms.

V. COMPLEX FRACTIONS

A. A complex fraction is one that contains a fraction in its numerator or denominator or in both.

B. A complex fraction is solved by simplifying the numerator and the denominator and dividing the first result by the second.

C. Review simultaneous linear equations of two and three unknowns.
  1. By elimination
  2. By substitution
  3. By graphs

Unit Four. Powers (Involution) (Four days)

I. VOCABULARY

Know the use and meaning of involution; base; exponent; odd and even powers.

II. MONOMIALS

A. The law of signs. Even powers produce positive, odd powers produce the sign of the given expression.

Examples: $(-2)^4 = 16$; $(2)^4 = 16$; $(-2)^3 = -8$; $(2)^3 = 8$

B. The law of coefficients. Raise the coefficient to the indicated power.

Examples: $(-2\ abc)^3 = -8a^3b^3c^3$; $(6d)^2 = 36d^2$

C. The law of exponents. Multiply the exponent of each factor by the indicated power, and write the product as the exponent of that factor in the result.

Examples: 1. $(-2a^2b^3)^4 = 16a^8b^{12}$

2. $$\left\{ \frac{-2ab^n}{3c^r} \right\}^3 = \frac{-8a^3b^{3n}}{27c^{3r}}$$

D. Find powers by the use of tables.

III. BINOMIALS

A. By actual multiplication find

$(3x-4y)^3$; $\left(\frac{x}{y} - \frac{y}{x}\right)^4$; $(2x \pm y)^2$

B. Learn and apply the laws of expansion used in the binomial theorem. These may be included in the following five.

1. Number of terms in the expansion
2. How to obtain the first term
3. Law of signs
4. How to obtain the coefficients
5. How to obtain the exponents

(See test in algebra)

In (1) above call the attention of the class to the fact that the coefficients of the expansion of odd powers, if graphed, arrange themselves in a plateau; whereas the coefficients of the even powers arrange themselves in a peak.

In (5) call the attention of the class to the fact that before simplifying the expansion, the sum of the exponents in any term should equal the given exponent.

NOTE: In the involution of a binomial, insist that in the first step the pupil should show only the expanded form of the binomial. In the second step he should complete the simplification.

IV. APPLICATION OF INVOLUTION TO PROBLEMS

1. Area of square

2. Area of circle
3. Men in army in square formation
4. Contents of a cube
5. Contents of a sphere
6. Other mathematical formulae involving powers

V. For Superior Pupils

Expansion of such expressions as $(a+b-c)^3$; $(-a-b-c)^4$

Suggested topic for report by superior pupil,

Pascal's triangle

Historical Reference. The binomial Theorem, The History of Mathematics, D. E. Smith, Ginn & Company.

(Sir Isaac Newton; Euler; Abel; Gauss)

Unit Five. Roots (Evolution) (Six days)

I. Vocabulary

Meaning and use of evolution; root; principal roots; index; surd; radical; radicand; radical sign.

II. Evolution of Monomials

A. Law of signs. Even roots produce the double sign ±. Odd roots the sign of the expression.

Examples: $\sqrt{9} = \pm 3$; $\sqrt[3]{-8} = -2$; $\sqrt[3]{8} = 2$

B. Law of coefficients: Extract the indicated root of the coefficient.

C. Law of exponents. Divide the exponent of each factor in the radicand by the index, and place the quotient as the exponent of that factor in the root.

Examples: $\sqrt[3]{-8a^3b^6} = -2ab^2$; $\sqrt{16a^8} = \pm 4a^4$

D. Teach the use of the tables of roots.

III. Evolution of Polynomials

A. Square roots of polynomials as

$$4x^4 + 20x^3 - 3x^2 - 70x + 49$$

B. Square roots of arithmetical numbers as

41616; 13; 3/5; .0065; 2; 3; 5; 7.

IV. Application of Evolution to Formulae and Problems

1. $V = \frac{4}{3}\pi r^3$ Solve for r.

2. $A = \pi r^2$ Solve for r.
3. Edge of cube. $V = e^3$ Solve for e
4. Hypotenuse of a right triangle. $c^2 = a^2+b^2$

V. TOPICS FOR SUPERIOR PUPILS

A. Develop $(a-b)^3$ by the binomial theorem, and from this result develop the method of finding the cube root.

B. The cube root of arithmetical numbers.

C. The cube root of polynomials.

D. The extraction of higher roots.

VI. HISTORICAL REFERENCE. The radical sign, History of Mathematics; D. E. Smith; Ginn and Company, pp. 407-409; Vol. II.

Unit Six. The Theory of Exponents (Ten days)

In previous work positive integral exponents have been used to show how many times the base is to be used as a factor. This definition cannot apply to an exponent that is negative, zero, or fractional.

I. VOCABULARY

Base; exponent; power; degree; root; factor.

II. LAWS OF EXPONENTS

Stress the memorization of the seven theories listed below with their applications.

A. Theory of exponents in multiplication

$a^m . a^n = a^{(m+n)}$; $(a^2 . b^3)\ (a^{2n} . b^{3n}) = a^{(2+2n)}\ b^{(3+3n)}$

B. Theory of exponents in division

$$\frac{a^m}{a^n} = a^{m-n}; \qquad \frac{a^4b^{5n}}{a^2b^{3n}c} = \frac{a^2b^{2n}}{c}$$

C. Theory of exponents in involution.

$(a^m)^n = a^{mn}$; $(3a^5)^n = 3^na^{5n}$

D. Theory of exponents in evolution

$\sqrt[n]{a^m} = a^{m/n}$; $\sqrt[2m]{a^{4m} . b^{6m}} = a^2b^3$

E. Theory of the zero exponent

Learn this proof

$\frac{a^m}{a^m} = 1$ Any number divided by itself gives 1

$\frac{a^m}{a^m} = a^0$ By theory (B) above

$\therefore A^0 = 1$. Any number with a zero exponent is therefore equal to 1.

F. Theory of the negative exponent.

Learn this proof

$\frac{a^m}{a^{3m}} = \frac{1}{a^{2m}}$ (Theory (B) above)

$\frac{a^m}{a^{3m}} = a^{-2m}$ (Theory (B) above)

$\therefore a^{-2m} = \frac{1}{a^{2m}}$

Hence any number with a negative exponent is equal to the reciprocal of that number with an equal positive exponent.

G. Theory of the fractional exponent.

$\sqrt[n]{a^m} = a^{m/n}$ Theory (D) above.

Therefore a number with a fractional exponent is equivalent to a problem in evolution, the denominator indicating the root, and the numerator the exponent.

The teacher should insist that the pupils master the seven theories, and be able to prove the theories for the zero and negative exponents.

III. USE OF EXPONENTS

Exponents are often used to write expressions in a shortened form:

$563 \times 10^8 = 56{,}300{,}000{,}000$

$563 \times 10^{-8} = \frac{563}{100{,}000{,}000} = .00000563$

In both cases above the last number is the same in value as the first, but more cumbersome to handle. Furthermore, it is not so easily compared with other numbers.

Examples of a few practical problems selected from a chemistry test:

a. $(54321)^{4/5} = ?$

b. Express 12300 as a power of 10

c. Express .000023 as a power of 10

d. Convert 1.23×10^3 to a whole number

e. Express 4.52×10^{-3} as a decimal

f. Add 3×10^{-3} to 4×10^{-4}

g. Multiply 4.5×10^{-3} by 6.6×10^{8}

h. $\dfrac{6.4 \times 10^{-11}}{3.5 \times 10^{-3}} = ?$

Unit Seven. Radicals (Thirteen days)

I. VOCABULARY

Radical; rational; irrational; surd; mixed surd; entire surd; radicand; index; imaginary; real number; similar radicals; rationalization; binomial quadratic surd; conjugate binomial quadratic surd; radical equation. The distinction of surd as opposed to rational, and real as opposed to imaginary should be stressed.

II. OPERATIONS INVOLVING RADICALS

A. Simplification of radicals. (Three processes)

B. Changing mixed surds to entire surds.

C. Changing radicals of different indices to radicals of the same index.

To be used in products and quotients.

D. Addition and subtraction of radicals. Stress the fact that only radicals of the same index and the same radicand may be added or subtracted.

E. Multiplication and division of radicals.

Change them to radicals of the same index before finding products or quotients.

F. Rationalization. The pupil should be taught to recognize a binomial quadratic surd and its conjugate. He should be taught to rationalize a (1) monomial radical denominator, (2) a denominator that is a binomial quadratic surd. At this time it will be well to stress again the square roots of 2, 3 and 5. Problems involving these should be given in order to obtain approximate results. The importance of rationalizing denominators should be thoroughly stressed and practiced.

G. Square roots of binomial quadratic surds. This should be taught in two ways.

1. By reversing the process of squaring a binomial quadratic surd as

Example: $\sqrt{30 + 12\sqrt{6}}$

Solution:

$$\begin{aligned} \sqrt{30 + 12\sqrt{6}} &= \sqrt{30 + 2\sqrt{216}} \\ &= \pm(\sqrt{18} + \sqrt{12}) \\ &= \pm(3\sqrt{2} + 2\sqrt{3}) \end{aligned}$$

2. By setting up and solving the equations

$$\sqrt{30+12\sqrt{6}}=\sqrt{x}+\sqrt{y} \quad (1)$$

$$\sqrt{30-12\sqrt{6}}=\sqrt{x}-\sqrt{y} \quad (2)$$

H. Radical equations. Explain the meaning of a radical equation.

Show the necessity of proof of the roots obtained. Explain extraneous roots. It is well to insist that the proof of the roots obtained is part of the major operation in the solution of all radical equations.

Unit Eight. Imaginary Numbers (Five days)

I. NECESSITY

In the solution of the equation $x^2 = -5$, it is required to find a number whose square is -5. No positive, negative, or rational number satisfies this condition, because the square root of any such number is positive. Extracting the square root of the given equation we have $x = \pm\sqrt{-5}$. Such a number as $\sqrt{-5}$ is called an imaginary number.

II. VOCABULARY

Imaginary number; complex number.

III. THE letter i is generally used to denote the imaginary unit $\sqrt{-1}$. Every imaginary number can be expressed as the product of a real number and the unit of imaginary. The number $\sqrt{-25}$ is written $5\sqrt{-1}$ or $5i$.

IV. POWERS OF THE IMAGINARY UNIT

$(i)^1 = i$

$(i)^2 = -1$

$(i)^3 = -i$

$(i)^4 = 1$

It is evident from the above that for every two minus signs under the radical we get one minus outside the radical

V. ADDITION AND SUBTRACTION

Imaginaries are added and subtracted as are radicals.

Example: $\sqrt{-2}+\sqrt{-8}-\sqrt{-50}=\sqrt{-2}+2\sqrt{-2}-5\sqrt{-2}$
$= -2\sqrt{-2}$ or $-2\sqrt{2}\,i$

VI. MULTIPLICATION AND DIVISION

It is advisable first to write the imaginaries in the form ai. The work is then performed as in the multiplication and division of radicals, i being treated as any other factor and subject to the substitution of the values of its powers.

Example: $$\frac{(\sqrt{-2})^3 \times (\sqrt{-8})^2}{(\sqrt{-32})^2} = \frac{2\sqrt{2}(-i) \times -8}{-32}$$
$$= \frac{-\sqrt{2}\,i}{2}$$

VII. GRAPHS

1. The graph of an imaginary number.
2. The graph of a complex number.
3. If the equation $x^2 - 4x + 5 = y$ be plotted the curve will not cut the x-axis.

   If $y = o$ and the equation is solved, x having imaginary values no real distance on the x-axis will be shown by the graph.

Supplementary topic. Imaginary numbers.

History of Mathematics; D. E. Smith; Vol. II, pp. 261 and 263.

Unit Nine. Quadratic Equations in One Unknown (Twenty-two days)

I. VOCABULARY

Know the use and meaning of the pure or incomplete quadratic equation, $x^2 = c$; the complete or affected quadratic equation, $ax^2 + bx + c = o$; the axioms of addition; subtraction; multiplication; division; powers; and roots.

II. THE PURE OR INCOMPLETE QUADRATIC

Procedure: Collect all the terms containing the square of the unknown to one member of the equation; extract the square root of each member

Example: $11x^4 - 176 = o$

$11x^4 = 176$

$x^4 = 16$

$x^2 = \pm 4$

$x = \pm 2$ or $\pm 2i$

Pure quadratics as applied to such formulæ as

$S = \frac{1}{2}gt^2$ (solve for t)

$A = \pi r^2$ (solve for r)

$a^2 + b^2 = c^2$

III. COMPLETE QUADRATICS

Every affected or complete quadratic equation must be reduced to the type form $ax^2 + bx + c = o$.

A. Solution by factoring.

B. Solution by completing the square.

C. Develop the formula from the general type.

D. Solution by use of the formula.

E. Solution by graph. Stress the difference of the graph. If the graph cuts the x-axis the roots are real and unequal; if the curve is tangent to the x-axis, the roots are real and equal, if it does not cross or touch the x-axis, the roots are imaginary.

Suggestion for getting the turning point. There are two ways of obtaining the coordinates of the turning point.

Method 1. Find half the sum of two values of x corresponding to a single value of y. Give x this value in the original equation and find y. This determines the coordinates of the turning point.

Method 2. Find the first derivative of the given equation; equate this to zero; substitute this value in the original equation and solve for y.

F. Teach checking by (1) substitution (2) sum and products of the roots without going into the theory of quadratics.

IV. Assimilative Material

A. From texts secure lists of application problems in which are equations and quadratic.

B. Supplementary work for superior pupils on this unit should be more difficult application problems as suggested under IV, A.

C. Historical reference. The quadratic equation. History of Mathematics; D. E. Smith, Vol. I, pp. 133, 134; 155-160. Vol. II, pp. 443-451, Ginn & Company.

Unit Ten. Equations in Quadratic Form (Optional)

I An equation is in the quadratic form when it conforms to the type form $ax^2 + bx + c = o$, namely that one term in which the unknown occurs must have an exponent twice that of the exponent of the other term in which the unknown occurs.

A review of the following topics would be worthwhile at this point.

a. Involution of binomials.

b. Product of conjugate surds.

c. Rationalization of denominators.

Whereas many equations in the quadratic form result in radical equations, the teacher should insist that the proof be considered a part of the solution.

ALGEBRA IV

Unit One. Equations in Quadratic Form (Ten days)

I. VOCABULARY

An equation in the form $ax^{2n} + bx^n + c = o$ is an equation in the quadratic form.

II. PROCEDURE IN SOLVING

Reduce the given equation to the type form $ax^{2n} + bx^n + c = o$ Solve for x^n by either the factor or formula method. From this result solve for x.

Unit Two. Properties of Quadratics (Five days)

I. VOCABULARY

Discriminant; character or nature of roots; signs of roots; rational; irrational; imaginary; real.

II.

A. Character or nature of roots of the equation $ax^2+bx+c=o$.

1. If the discriminant $b^2-4ac>o$, the roots are real and unequal; rational if $\sqrt{b^2-4ac}$ is rational; irrational if $\sqrt{b^2-4ac}$ is irrational.
2. If $b^2-4ac=o$, the roots are real, equal, rational.
3. If $b^2-4ac<o$, the roots are imaginary.

B. Law of signs of the roots.

1. If c is positive both roots have the same sign and this sign will be opposite the sign of b.
2. If c is negative the roots will have opposite signs, the numerically greater having the sign opposite that of b.

C. The sum of the roots is equal to $\frac{b}{a}$ with the sign changed.

The products of the roots will be $\frac{c}{a}$ with the same sign.

D. Determining the value of the literal coefficients as:

1. Find the value of m in $8x^2+mx+18=o$ that will make the equation have unequal roots.
2. Find the value of m that will make the roots of $(m+2)x^2-20x+m=o$ real and equal.

3. Find the value of m that will make the roots of $mx^2+4mx+50=o$ imaginary.

E. To construct the equation if the roots are given.

1. By the use of the sum and product of the given roots.
2. By the factor method.

Unit Three. Maxima and Minima (Five days)

I. VOCABULARY

Maximum and minimum; function; $f\ (x)$; $f'(x)$; $f''(x)$

II.

To solve graphically such equations as $ax^n+bx^{n-1}+cx^{n-2}..+h=y$

Teach. How to find the first and higher derivatives of the typical equation using examples of the second and third power equations. It is useful to show how the turning points (maximum and minimum points) in these equations may be found.

Three procedures for doing this follow:

A. To find the value of x which gives the minimum values of $f(x)$, and also to determine this minimum value.

Procedure. Set $f\ (x)=y$ say $x^2-5x+6=y$

$$\text{Then } x^2-5x+(6-y)=o$$

Let the discriminant $=o$. This will give the value of y. By substituting this in the given equation, the value of x will be found.

In the general equation $ax^2+bx+c=y$ where a is positive we find that the coordinates of the minimum point of the curve are

$$x=\frac{-b}{2a}; \qquad y=\frac{-b^2-4ac;}{4a}$$

If y is positive, the minimum point of the curve is positive and the equation will have imaginary roots, for it will not touch the x-axis. If a is negative the graph will be inverted and the above procedure will give the maximum point of the curve.

B. By the use of the derivative.

It is proved in the calculus that where $f'\ (x)=o$ and $f''(x)<o$ there is a maximum value of y; and where $f'(x)=o$ and $f''(x)>o$ there is a minimum value of y.

Procedure 1. Given $f(x)=2x^3-3x^2-12x+12$

2. Find $f'(x)=6x^2-6x-12$

3. Set $f'(x)=o$ and solve for real roots to find the critical value $x=2, -1$

4. Find $f''(x)$ or $f''(x)=12x-6$

5. Substitute each critical value in ascending order, for the variable in the second derivative $f''(x)$. If the result is negative, then $f(x)$ is a maximum for that critical value; if the result is positive, the function is a minimum for that critical value.

C. Procedure.

By the use of the first derivative $f'(x)$ the critical values may be found as in B above. Then by increasing or decreasing these critical values by $\pm h$, we may find whether the direction of the curve changes from + to —, or from — to +. If the former, this is a maximum value; if the latter, it is a minimum value.

NOTE:—Methods B or C are preferable when $f(x)$ is of the third degree or higher. For an equation of the second degree use method A.

Unit Four. Systems of Quadratics (Twenty days)

SIMULTANEOUS QUADRATIC EQUATIONS

I. REVIEW (a) Simultaneous Linear Equations

(b) Quadratic Equation $ax^2+bx+c=o$

II. VOCABULARY Homogeneous equations; symmetrical equations; homogeneous quadratic equations.

III. TYPES There seem to be three outstanding types of simultaneous quadratic equations which should be thoroughly understood by the student. It then becomes a simple matter to solve other forms of simultaneous quadratics.

TYPE A. $x+y=7$

$3x^2+y^2=43$

One equation is simple, the other quadratic.

Procedure: Find one unknown in terms of the other from the simple equation; substitute this value in the quadratic; solve this quadratic equation; resubstitute in the simple; test the values obtained.

TYPE B. $x+y=7$

$xy=10$

If the equations are symmetrical.

Procedure: Find the values of $x+y$, xy, $x-y$; treat the equations containing $x+y$ and $x-y$ simultaneously; group the results correctly; test in both equations.

TYPE C. $x^2-xy+y^2=21$

$y^2-2xy=-15$

If each equation is quadratic and homogeneous in the unknown terms.

Procedure: Eliminate the known terms from the equations; solve the resulting quadratic equation for x in terms of y; substitute these values in one of the given equations; solve; test by substituting in the other original equation.

Numerous examples of application problems producing simultaneous quadratics should be stressed.

IV. GRAPHIC METHOD

The following types should be taught.

(1) $ax+by=c$ produces a straight line

(2) $ax^2+bx+c=o$ produces the parabola

(3) $ax^2+by^2=c$ produces the ellipse

(4) $x^2+y^2=c$ produces the circle

(5) $\frac{x^2}{a^2}-\frac{y^2}{c^2}=c$ produces the hyperbola

(6) $xy=c$ produces the hyperbola

The student should be made aware of the fact that in simultaneous equations the values obtained should satisfy both equations. When solved graphically the points of intersection of the graphs give the roots which satisfy the given equations.

Unit Five. Ratio, Proportion, and Variation (Five days)

RATIO AND PROPORTION

I. VOCABULARY Ratio; proportion; antecedent; consequent; mean proportional; third proportional; fourth proportional; inverse ratio; continued proportion; series of equal ratios; means; extremes.

Explain that a ratio is an indicated division where the antecedent is the dividend or numerator, and the consequent is the divisor or denominator.

It follows therefore that a ratio is commonly written as a fraction.

A ratio may be written $\frac{a}{b}$ or $a:b$. A proportion may be written $\frac{a}{b}=\frac{c}{d}$ or $a:b=c:d$

II. PRINCIPLES OF PROPORTION

If four numbers are in proportion

1. The product of the extremes equals the product of the means.
2. They are in proportion by alternation
3. By inversion
4. By composition
5. By division
6. By composition and division
7. Their like powers or roots are in proportion
8. In a continued proportion the means is equal to the square root of the product of the extremes
9. In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.
10. If four numbers are in proportion and the antecedents are equal then the consequents are equal.
11. If the product of two numbers equals the product of two other numbers, then either set of factors may be made the extremes and the other two the means. The test of a proportion is to see if the product of the extremes is equal to the product of the means.

The student should be able to prove the eleven principles listed above and acquire facility in their applications.

VARIATION

Variation; direct variation; inverse variation; joint variation; sign of variation $\propto$.

Explain that if $x \propto y$ then the equation $\frac{x}{y} = m$ may be written.

Likewise if $x \propto \frac{1}{y}$ then $xy = m$. In making clear the meaning of variation show its practical use in proportion; in Charles' and Boyle's laws; in intensity and distance of light; in laws of gravitation; in price of commodities; in time, rate, and distance; in principal, interest, rate and time; in altitude and pressure; in time and work; in various area and volume formulae. The functional relation should be stressed at this point. The symbol $f(x)$ to represent an equation involving x should again be stressed and explained.

Unit Six. Logarithms (Fifteen days)

I. VOCABULARY

Logarithm; exponent; base; characteristic; mantissa; antilogarithm;

cologarithm; tabular difference; interpolation; common system of logarithms. The theory of logarithms must be connected with the theory of exponents to have meaning to the student.

II. DEVELOPMENT AND FUNDAMENTAL MATERIALS

A. Since a logarithm is an exponent and since $2^3 = 8$, then 3 is the logarithm of 8 on base 2. This is usually written $\log_2 8=3$. From this develop 10^1, 10^2, 10^3 etc. showing why the characteristic is 1 less than the number of digits that precede the decimal point.

Then $\left(\frac{1}{10}\right)^1$, $\left(\frac{1}{10}\right)^2$, $\left(\frac{1}{10}\right)^3$, or .1, .01, .001 will show why the characteristic of a number less than one is negative. Review negative exponents as $10^{-1} = \frac{1}{10}$; therefore the logarithm of .1 is —1. Review the zero exponent. If $10^\circ = 1$ then $\log 1 = o$.

B. Drill on finding the characteristic of a number greater than 1.

C. Drill on finding the characteristic of a number less than 1.

D. Negative numbers do have logarithms.

E. Practice reading mantissas from the table.

F. Practice finding the antilogarithm of a logarithm.

G. Interpolation

H. The logarithm of $0 = \propto$

III. USES OF LOGARITHMS

A. Multiplication by use of logs.

B. Division by use of logs and cologs.

C. Involution by use of logs.

D. Evolution by use of logs.

Review again the theory of exponents and show that the rules governing exponents also govern logarithms.

IV. EXPONENTIAL EQUATIONS

Develop this through simple equations as $2^x = 16$ find x. Show the need for a solution not easily obtained, as $31 = 2^x$.

Amount at compound interest $A = P \ (1 — r)^n$ solve for n

V. GRAPH SIMPLE EXPONENTIAL EQUATIONS

VI. PRACTICAL APPLICATIONS

A. Compound interest

B. Compound amount

C. Annuities.

VII. ADDITIONAL TOPICS

A. Common logarithms changed to some other base than 10.

B. Historical reference. The History of Mathematics, D. E. Smith; Vol. 1 pp. 338, 391, 394, 416, 433, 544, 436, 534. Vol. 2 p. 513.

Unit Seven. Arithmetic and Geometric Progressions (Five days)

I. ARITHMETIC PROGRESSION

A. Vocabulary.

Series; progression; common difference; arithmetic mean.

B. Derive the formula for any term. An arithmetic series is usually written $a, a+d, a+2d \ldots l$.

C. From the formula $l = a + (n-1)d$ solve for a,n,d.

D. Derive the formula for S. $S = \frac{n(a+l)}{2}$

Derive the formula for s, substituting $a + (n-1)d$ for l in the formula for s given above.

E. The arithmetic mean between two numbers is found by the formula

$$M = \frac{a+b}{2}$$

F. Apply the formulae to drill and practical problems.

G. Stress the memorization of the four important formulae used in A.P.

II. GEOMETRIC PROGRESSION

A. Vocabulary.

Common ratio; geometric mean;

B. A geometric series is usually written $a, ar, ar^2 \ldots l$.

Show how to find r if two successive terms in a geometric series are known.

C. Derive the formula $l = ar^{(n-1)}$ and be able to solve for a,r,n.

Review logs to find n.

D. Derive formulae for sum S.

$$S = \frac{a - rl}{1 - r}$$

$$S = \frac{a - ar^n}{1 - r}$$

E. Give drill and practical problems on geometric progressions.

F. Infinite geometric series

Develop the idea of the infinite series.

From the formula $S = \frac{a - rl}{1 - r}$

Show how for an infinite series this becomes $S = \frac{a}{1 - r}$ as l approaches zero.

G. Historical reference. The History of Mathematics, D. E. Smith. Volumes I and II. Ginn and Company.

Unit Eight. Binomial Theorem (Five days)

I.

By actual multiplication show that

$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$

$(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$

Then call attention to the following facts:

1. The product has one more term than the given exponent.
2. The first term in the expansion is the same as the first term in the binomial raised to the given power.
3. If the binomial is the sum, all signs in the expansion will be plus; if the binomial is the difference, the terms will be alternately plus, minus.
4. The coefficient of any term may be found by multiplying the coefficient of the preceding term by the exponent of first factor in that term and dividing by the number of terms already found.
5. The exponent of the first term in the binomial decreases 1, whereas the exponent of the second term of the binomial increases 1 in each succeeding term. The sum of the exponents in any term shall equal the given exponent.

II.

PROOF BY INDUCTION

$$\text{Teach } (a+b)^n = a^n + \frac{n}{1} a^{(n-1)} b + \frac{n(n-1)}{1 \cdot 2} a^{(n-2)} b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{(n-3)} b^3 \ldots\ldots \text{etc.}$$

From this the following laws may be induced.

In any term, say the rth term.

A. The exponent of b is $(r-1)$.

B. The exponent of a is $(n-r+1)$.

C. The factors in the numerator of the coefficient will begin with n and end with $(n-r+2)$.

D. The factors in the denominator will be 1^{r-1} or $(r-)!$.

III. Drill and practice problems should include the following types:

A. Expansion of $(a \pm b)$ raised to odd or even integral powers.

B. Expansion of $(a \pm b)^{2/3}$ to any number of required terms.

C. Finding any desired term directly.

D. Finding a term that does not contain the literal factors.

E. Compound interest in which $A = P(1+r)^n$.

IV. SUGGESTED ADDITIONAL TOPICS

A. Find square root by binomial theorem.

B. The normal curve.

C. Proof of the validity of the binomial theorem for positive integral exponents.

Unit Nine. Numerical Trigonometry of the Right Triangle (Ten days)

I. VOCABULARY

Trigonometry; trigonometric functions; sine; cosine; tangent; angle of depression; angle of elevation; table of natural functions; solving a right triangle.

II.

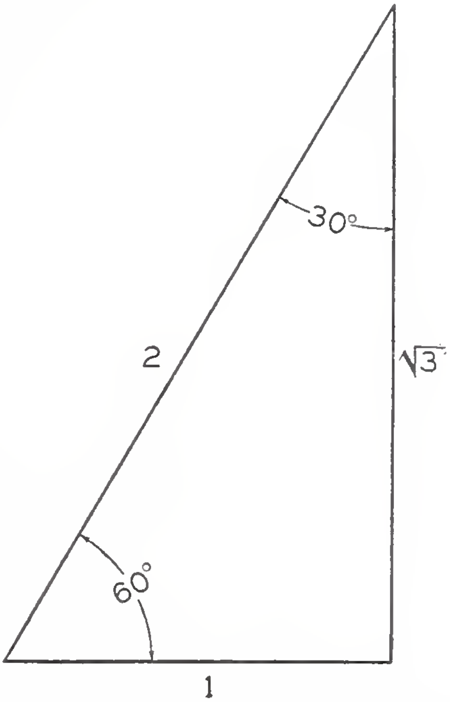
A. The trigonometry of the 30° right triangle.

Review the theorem of Pythagoras and let the student derive and memorize the following:

$$\sin 60 = \frac{\sqrt{3}}{2} = \cos 30$$

$$\cos 60 = \frac{1}{2} = \sin 30$$

$$\tan 60 = \sqrt{3} = \cot 30$$



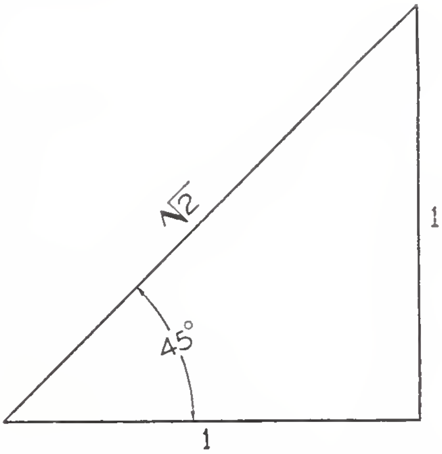
Stress the fact that a function of an angle is equal to the cofunction of its complement.

B. The trigonometry of the 45° right triangle.

Derive and memorize:

$$\text{Sine } 45 = \frac{1}{\sqrt{2}} = \cos 45$$

$$\text{Tan } 45 = 1 = \cot 45$$



C. The trigonometry of any right triangle using sines, cosines, and tangents as found in the tables.

III.

The solution of the right triangle when given

1. Two sides.
2. One side and one acute angle.

IV.

Historical reference: Hipparchus; Regiomontamus; Development of trigonometry in the Seventeenth Century. History of Mathematics, D. E. Smith; Vols. 1 and 2.

Additional Units

If time permits, the following topics should be included:

1. Determinants.
2. Theory of equations.
3. Permutations and combinations.
4. Slide rule.

UNIT TESTS—ALGEBRA III.

Test on Unit One. Fundamental Operations—Part One

Test I (40 minutes)

1. The ——— is the number from which another number is subtracted.
2. The ——— is the number to be subtracted.
3. An axiom is ———.
4. A term is an algebraic expression whose parts are not separated by ——— or ———.
5. Expressions which contain the same letters with same exponents are ——— terms.
6. An expression of one term is a ———.
7. An expression of more than one term is a ———.
8. An expression of two terms is a ———.
9. An expression of three terms is a ———.
10. The value of the unknown in an equation is called the ———.
11. Underline correct form: $5a = (a \cdot a \cdot a \cdot a \cdot a)$ $(a+a+a+a+a)$.
12. Underline correct form: $a^5 = (a \cdot a \cdot a \cdot a \cdot a)$ $(a+a+a+a+a)$.
13. The *5* in question 11 is ———.
14. The *5* in question 12 is ———.

15. Express 3a divided by 4b in three ways, using algebraic symbols. ———— ———— ————.

16. A literal equation used to state a principle or rule is called a ————.

Underline correct sentences in following questions:

17. In a series of additions, subtractions or both
    (1) operations must be performed in order from left to right.
    (2) operations may be performed in any order.

18. In a series of multiplications
    (1) operations must be performed in order from left to right.
    (2) operations may be in any order.

19. In a series of divisions, or of multiplications and divisions.
    (1) operations must be performed in order from left to right.
    (2) operations may be in any order.

20. In a series involving addition, subtraction, multiplication, and division.
    (1) operations must be performed in order from left to right.
    (2) operations may be in any order.
    (3) addition and subtraction performed first, then multiplication and division last.
    (4) multiplication and division first, then addition and subtraction last.

Fundamental Operations—Part Two

Express by use of Algebraic symbols:

(1) One half the sum of two numbers ————

(2) The difference of the squares of two numbers ————

(3) The sum of the cubes of two members ————

(4) If a and b are the dimensions of a rectangle, represent the area ————, its perimeter ————

(5) What is the effect on area of the rectangle of example 4 if a is doubled and b tripled ————

(6) Represent the difference in area between two squares whose sides are $x + 1$ and x respectively ————

(7) If a is one of two numbers, whose sum is b, what is the other number? ————

(8) What is the value in cents of x dimes? __________ Of r quarters? __________

(9) A dealer gained r dollars by selling goods for s dollars. How much did they cost him? __________

(10) B walked r miles and C walked s miles farther. How many miles did C walk? __________

Test on Unit Two. Factoring

TEST I (40 minutes)

Factor if possible:

1. $14ab^4 - 35\,a^3b^2$
2. $ab - ay + bx - xy$
3. $25a^2 - 5ab + b^4$
4. $a^5 + b^5$
5. $x^8 + 4y^8$
6. $9x^4 - 3x^2 + 4$
7. $4x^2 - 12xy + 9y^2 - x^2 + 2xy - y^2$
8. $24 + 5x - x^2$
9. $(a^2 + 7a)^2 + 20(a^2 + 7a) - 96$
10. $A^6 + b^6$
11. $x^4 + x^3 - 3x^2 - 17x - 30$
12. $A^5 - b^5$
13. $4x^4 - 6y^2 - 9 - y^4$
14. $64 - 64a^3 - a^6 - a^9$
15. $a^2 - .7a + 1$

TEST II (40 minutes)

Factor if possible:

1. $2\,ay^3x^2 - 8ayx^4$
2. $ax - 9x + ay - 9y$
3. $x^7 + y^7$
4. $x^3 - 6x^2 + 10x - 3$
5. $x^{10} + y^{10}$
6. $x^{10} - y^{10}$
7. $a^2 - c^2 + b^2 - d^2 - 2cd - 2ab$
8. $x^2 - y^2 - 2yz - z^2$
9. $x^3 - 7x - 6$
10. $(2a - 3)^2 - 3(2a - 3) - 10$
11. $a^3 - 8$
12. $a^4 - a^2\,b^2 + 16\,b^4$
13. $a^3 + b^3 - a - b$
14. $x^4 + 4$
15. $x^4 - 2x^2 + 1$

Test on Unit Three. Fractions

TEST I (40 minutes)

I. State the principle involved in the following:

$$\frac{b - a - c}{c - a - b} = \frac{a - b + c}{a + b - c}$$

II. Reduce to a single fraction: $c - 2 - \dfrac{c - 5}{c - 4}$

III. Reduce to lowest terms: $\dfrac{ab^2 - ac^2}{ab^2 - abc}$

IV. Simplify: $\dfrac{1}{2(x - 1)} - \dfrac{4}{x - 2} + \dfrac{7}{2(x - 3)}$

V. Simplify: $\dfrac{x + 1}{x + 5} + \dfrac{x - 2}{x - 3} - \dfrac{2x^2 + x - 13}{x^2 + 2x - 15}$

VI. Simplify: $\dfrac{a^3 - x^3}{a^2 + 2ax + x^2} \times \dfrac{5x}{3x - 3a} \times \dfrac{6a + 6x}{5a^2 + 5ax + 5x^2}$

VII. Simplify: $$\frac{\dfrac{a}{1+a} + \dfrac{1-a}{a}}{\dfrac{a}{1+a} - \dfrac{1-a}{a}}$$

TEST II (40 minutes)

I. State the principle involved in: $$\frac{(4-b)\ (a-3)}{(b-5)\ (7-a)} = \frac{(b-4)\ (a-3)}{(b-5)\ (a-7)}$$

II. Reduce to a single fraction: $$\frac{3a}{x^2yz} + \frac{2a}{x^2y^4z^5} - \frac{6a}{xy^6z}$$

III. Reduce to lowest terms: $$\frac{2a^2 + 19a + 35}{3a^2 + 15a - 42}$$

IV. Simplify: $$\frac{x-2}{2} - \frac{x}{x+2} - \frac{x^2-2x}{2(x-2)}$$

V. Simplify: $$\frac{2x}{x-1} - \frac{6}{x+3} - \frac{8x}{3-2x-x^2}$$

VI. Simplify: $$\frac{x^2-y^2}{x^2-3xy+2y^2} \times \frac{xy-2y^2}{x^2+xy} \times \frac{x^2-xy}{(x-y)^2}$$

VII. Simplify: $$\left\{x-2+\frac{1}{x+2}\right\} \div \left\{x+2+\frac{1}{x-2}\right\}$$

VIII. Simplify: $$\left\{5-\frac{a^2-19x^2}{a^2-4x^2}\right\} + \left\{3-\frac{a-5x}{a-2x}\right\}$$

Test on Unit Four. Powers

TEST I (40 minutes)

1. $(2a^2)^2$
2. $(-2ab)^2$
3. $(-a^{-1})^2$
4. $(3a^3)^3$
5. $(-3a^4)^3$
6. $\left\{-\frac{2a}{3b}\right\}^3$
7. $(-2)^5$
8. $(-2)^6$
9. $(a^n)^2$
10. $(-2a^{n-2})^3$

By the use of the binomial theorem, expand the following:

11. $(x+y)^8$
12. $(x-2)^4$
13. $(2a-3)^5$
14. $\left\{\frac{x}{y}-\frac{y}{2}\right\}^5$
15. The number of terms in any expansion equals ....
16. The sum of the exponents in any term of the expansion equals ....
17. In the formula $V = \frac{4}{3}\pi r^3$, find V if $r = 5$.
18. Which term of the expansion of $\left\{x^2-\frac{1}{x}\right\}^6$ does not contain x after simplification?

Test on Unit Five. Roots

TEST I (40 minutes)

1. What is the square root of 36?
2. What is the cube root of 8?
3. What is the cube root of -8?
4. What is the square root of -36?
5. What is the square root of 4489?
6. What is the square root of 6?
7. What is the square root of 2.875?
8. What is the square root of $4x^4 - 4x^3 - x^2 + x + \frac{1}{4}$?
9. What is the cube root of $-8a^3b^6$?
10. What is the fifth root of $32a^5$?
11. What is the fourth root of 625?
12. The fourth root may be found by ....
13. What is the cube root of $64a^6b^9$?
14. The volume of a cylinder equal πr^2h. If $V = 216$, $h = 31.62$, $\pi = \frac{22}{7}$, find r.

Test on Unit Six. Theory of Exponents

TEST I (40 minutes)

Express with positive exponents:

1. $x^{-4}y^3$
2. $a^{5/2}b^{-8}$
3. $m^{-1/4}n^{1/6}$
4. $3n^{-1}x$
5. $a^{-2}m^{-3}$

Transfer all literal factors from the numerator to the denominator.

6. $\frac{4a^7}{3}$
7. $m^{-2}n^{-3}$

Transfer all literal factors from the denominator to the numerator.

8. $\frac{1}{x^8}$
9. $\frac{3}{ax^{-4}}$

Simplify:

10. $x^{-2} \cdot \sqrt[4]{x^9}$

11. $x^5 \div x^9$

12. $\sqrt{a^5} \div \sqrt[4]{a^{-8}}$

13. $(2n^{-3/4} - 5 - 6n^{3/4}) \cdot (3n^{-3/4} - 4)$

14. $(a^{-1} - b^{-1}) \div (a^{-1/3} - b^{-1/3})$

15. $(a^3)^{-4}$

16. $(a^{-5})^{-7/5}$

17. $(-32)^{4/5}$

18. $x^0 \cdot 243^{-6/5}$

19. $(\frac{1}{2})^0$

20. $8^{2/3} + 25^{-1/2} + \dfrac{1}{4^{-1}}$

TEST II (40 minutes)

Simplify:

1. $8^{1/3}$

2. $\dfrac{a^0b^2}{b^{-2}}$

3. $\dfrac{x^m}{x^n} \cdot \dfrac{x^{-n}}{x^0}$

4. $x^{2/3} \div x^{5/3}$

5. $x^0 = 1$. Prove

6. $a^{1/3} \cdot a^{1/2} \cdot a^{-1/3}$

7. $x^0 \cdot x^4 \cdot x^{-4}$

8. $42t^{-3} \div t^2$

9. $x^{1-n} \cdot x^{n-1}$

10. $x^2 \div \dfrac{1}{x^2}$

11. $(x^{-2} - 1)\ (x^{-2} + 3)$

12. $10^{-2} \cdot 100$

13. $81^{-1/2} + 9^{3/2} + 32^0$

14. $(a^{-2} + a^{-1}b + b^2) \div (a^{-1} + a^{-1/2}b + b)$

15. $(a + b)^0 \cdot (a + b)^2$

16. $x^{-3n-2} \cdot x^{2n-3}$

17. $\sqrt[4]{16^{-1}} \cdot 32^{-1/5} \cdot 4^{3/2}$

18. $\sqrt[3]{\sqrt{(16^{-6})^2}}$

19. Solve $x^{2/3} = 4$

Test on Unit Seven. Radicals

TEST I (40 minutes)

Simplify:

1. $\sqrt{8x^3}$

2. $\sqrt[3]{27xy^3}$

3. $\sqrt[3]{\frac{1}{9x^2}}$

4. $\sqrt{72} - \sqrt{32}$

5. $\sqrt[3]{24x} - \sqrt[3]{54x}$

6. $\sqrt{3} \cdot \sqrt{6}$

7. $2\sqrt{7x} \cdot 3\sqrt{14x}$

8. $(\sqrt{2} + x)\ (\sqrt{2} - x)$

9. $\frac{12\sqrt{20}}{3\sqrt{5}}$

10. $\frac{9}{\sqrt{6}}$

11. $\frac{2\sqrt{6} \cdot \sqrt{12}}{9}$

12. $\frac{3 + \sqrt{6}}{3 - \sqrt{5}}$

13. $\sqrt{11 - \sqrt{72}}$

14. $\sqrt{a^3} + \sqrt[4]{a^2} + 3\sqrt[6]{a^3}$

15. $\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}$

16. $8\sqrt{x} \div 2\sqrt[3]{y}$

17. Solve $\sqrt{x} = 3$

18. $\sqrt[3]{x} = -2$

19. $3\sqrt{x} - 6 = 0$

20. $5\sqrt{10 - x} + 2 = 12$

21. $\frac{3}{\sqrt{x - 4}} + \sqrt{x + 4} = 4$

TEST II (40 minutes)

Simplify:

1. $\sqrt{36x^2(a + 1)^3}$

2. $\sqrt{a^2 + \frac{a^2}{4}}$

3. $\sqrt{2^2 \cdot 3} + \sqrt[4]{2^4 \cdot 3^2} - \sqrt[6]{2^6 \cdot 3^3}$

4. $\frac{1}{\sqrt{5} - \sqrt{7}} + \frac{1}{\sqrt{5} + \sqrt{7}} + 2\sqrt{5}$

5. $\frac{5\sqrt{6} - 7\sqrt{3}}{2\sqrt{3}}$

6. $\sqrt{\frac{1}{3}} \div \sqrt[8]{\frac{1}{3}}$

7. $\frac{2\sqrt{a} - \sqrt{b}}{\sqrt{a} + 3\sqrt{b}}$ Check by letting $a = 9$ and $b = 4$

8. If $A = \sqrt{s(s - a)\ (s - b)\ (s - c)}$ is the formula for the area of a triangle whose sides are a, b, and c, and $s = \frac{1}{2}$ the perimeter, find by this formula the area of a triangle whose sides are 6, 8, and 12.

9. A rectangle is formed by placing three squares each 6″ on a side, adjacent to each other. Find its diagonal.

10. Find the value of $2\sqrt{12} - 3(27)^{1/2} + \sqrt{75} - 2(108)^{1/2}$

TEST III (40 minutes)

Solve and check:

1. $2\sqrt{x} - 3 = \sqrt{x} + 2$
2. $\sqrt{x + 1} = a - 1$
3. $\sqrt{x - 1} + \sqrt{x} = \sqrt{4x + 1}$
4. $\sqrt{x - 5} + \sqrt{x + 8} = \sqrt{x - 3} + \sqrt{x + 2}$
5. $\sqrt{x + 5} + \sqrt{x} = \frac{15}{\sqrt{x + 5}}$

TEST IV (40 minutes)

Solve and check:

1. $\sqrt[3]{x^3 + 3x^2} = x + 1$
2. $\sqrt{x + 12} = \sqrt{x - 3} + 3$
3. $\sqrt{x + 1} + \sqrt{x + 2} = \sqrt{x - 1} - \sqrt{x - 3}$
4. $\sqrt{x - 5} + \sqrt{x + 7} = 6$
5. $\frac{21}{\sqrt{x - 7}} - \sqrt{x - 7} = \sqrt{x}$

TEST V (40 minutes)

Solve and check:

1. $\sqrt[4]{x - 1} = \sqrt{5}$
2. $\sqrt{x^2 - 9} + x = 9$
3. $2(\sqrt{x} + 5)\ (\sqrt{x} - 5) = -35$
4. $\sqrt{\frac{7}{4}} + x = \sqrt{x} + \frac{1}{2}$
5. $\frac{2\sqrt{x} - 14}{\sqrt{x} - 6} = \frac{2\sqrt{x} - 2}{\sqrt{x} + 2}$

TEST VI (40 minutes)

Solve and check:

1. $\sqrt[3]{5x + 2} = 3$
2. $\sqrt{7 + x} = 7 - \sqrt{x}$
3. $\sqrt{x - 16} + \sqrt{x} = 8$
4. $\sqrt{x} - \frac{36}{\sqrt{x - 9}} = -\sqrt{x - 9}$
5. $\frac{\sqrt{5 + x} - 5\sqrt{5 - x}}{\sqrt{5 + x} + 5\sqrt{5 - x}} = \frac{1}{2}$

Test on Unit Eight. Imaginaries

TEST I (40 minutes)

Find the sum of:

1. $3(-1/9)^{\frac{1}{2}} + 5(-1/5)^{\frac{1}{2}} - 2(-4/9)^{\frac{1}{2}} + 6(-1/81)^{\frac{1}{2}}$

2. $c\sqrt{-1} + \sqrt{-16c^2} + \sqrt{-c^2} - \frac{2}{c}\sqrt{-c^4}$

Simplify:

3. $(3\sqrt{2} - 2\sqrt{-1})^3$

4. $\dfrac{\sqrt{c} + 3\sqrt{-2c}}{\sqrt{c} - \sqrt{-c}}$

5. $\dfrac{i+1}{i-1} + \dfrac{i-1}{i+1}$

6. Plot the points:

   $3i; -2i; 2+3i; -2-3i; -2+3i$

7. Define an imaginary number.

8. Find the value of: $\dfrac{1}{2+i} + \dfrac{1}{5} + \dfrac{1}{2-i}$

Test II (40 minutes)

1. Multiply $(\sqrt{-1} - \sqrt{-2} - \sqrt{-3})(\sqrt{-1} + \sqrt{-2} - \sqrt{-3})$

2. What famous mathematic's name is connected with imaginary numbers?

3. Simplify $\left\{\dfrac{1}{3+i} + \dfrac{1}{3-i}\right\}\left\{\dfrac{1}{3+i} - \dfrac{1}{3-i}\right\}$

4. Find the value of: $(a - \sqrt{b})\ (a + \sqrt{b})\ (a - \sqrt{-b})\ (a + \sqrt{-b})$ if $a = 2$, $b = 3$

5. Plot $\pm 5i$; $\pm 5 - 3i$; $5 + 3i$

6. Simplify $\dfrac{i^6 - i^5 + i^4}{i\sqrt{2}}$

Test III (40 minutes)

1. Write the rationalizing factor of $\sqrt{-3}$; $(2 - \sqrt{-3})$

2. Rationalize the expressions in 1, above.

3. Simplify $\dfrac{(2+i)^2 + (2-i)^2}{3i}$

4. Expand and simplify $(3i - 2i)^4$

5. Multiply $(-2i - i\sqrt{2})\ (3i + 2i)$

6. Plot $5 \pm 2i$; $-5 \pm 2i$; $4i$

Test on Unit Nine. Quadratic Equations

Test I (40 minutes)

1. Write the type form of (a) the incomplete quadratic; (b) the complete quadratic.

2. Solve by completing the square $x^2 + 5x + (-\frac{11}{4}) = 0$
3. Solve by the factor method
4. Solve by formula $\frac{3}{x - 5} - 5 = \frac{-2x}{x - 3}$
5. A man bought two farms for $4800 each. One contained 20 acres more than the other and cost him $8.00 less per acre. How many acres were there in each farm?

TEST II (40 minutes)

1. Name the graph (curve) produced by the equation

$$ax^2 + bx + c = 0$$

Abscissae are measured along which axis? In which direction are the quadrants named? What are the signs of points in quadrants 2 and 4?

2. Solve by graphs and note the values of the unknown in each case.

   a. $4x^2 - 12x + 9 = 0$

   b. $2y^2 + 3y - 9 = x$

   c. $x^2 - 4x + 5 = y$

3. Without graphing find the turning point of the equation

$$3x^2 - 5x + 7 = y$$

UNIT TESTS ALGEBRA IV

Test on Unit One

TEST I Equations in quadratic form (40 minutes)

1. When is an equation in the quadratic form?
2. Solve and prove:

   a. $x+2\sqrt{x} = 15$

   b. $(2x^2+7) + 2\sqrt{2x^2+7} = 35$

   c. $8x^{-3}+19x^{-3/2} - 27 = 0$

   d. $x^2 - 5x+4 = 10\sqrt{x^2-5x-5}$

TEST II

Solve and check:

a. $3x^{-2/3} - 7x^{-1/3} - 6 = 0$

b. $x^6 - 7x^3 - 8 = 0$ (Two roots only)

c. $6y - \sqrt{6y+4} = 52$

d. $\frac{x}{5} + \frac{\sqrt{x}}{6} - \frac{1}{11} = 0$

e. $\frac{x^2+3}{12x} + \frac{12x}{x^2+3} = \frac{-10}{3}$

TEST III

Solve and check:

a. $3x^{1/4} + 10x^{1/2} - 13 = 0$

b. $(x-5)^2 + 5(x-5) = 6$

c. $x^{-2} + 2x^{-1} = 8$

d. $(x^2+x) + 3\sqrt{x^2+x+6} = 48$

e. $(3x^2-2x)^{1/3} = 2$

Test on Unit Two. Properties of Quadratics (40 minutes)

1. Without solving fill in the table below

Problem	*Discriminant*	*Nature of Roots*	*Signs of Roots*	*Product of Roots*	*Sum of Roots*
a. $3x^2+15x-12 = 0$					
b. $\frac{1}{x} = 1-x$					
c. $9x-7-3x^2 = 0$					

2.

a. For what value of k will the roots of $y^2-ky+6\frac{1}{2}=0$ be real and equal?

b: For what value of m will the roots of $x^2-(m-3)x+2m-9=0$ be real and equal?

c. For what value of t will the roots of $tx^2+4x+1 = 0$ be real and equal?

3. Make the equations whose roots shall be:

(a) $\pm\sqrt{2}$; (b) 0,0; (c) $-2 \pm \sqrt{6}$; (d) $2 \pm \sqrt{-3}$; (e) -3, .04

4. Find the value of R so that 3 will be a root of the equation

$\frac{R}{x} - x = 6$

5. If $3x-1$ is a factor of $kx^2-13x-6 = 0$, find k.

6. Form the equation the sum of whose roots shall be 3 times the product of the roots of $3x^2-5x = 6$; and the product of whose roots shall be twice the sum of the roots $\frac{x^2}{2} - 3x = 7$

7. Find k in $x^2 + kx - 6 = 0$ if one root exceeds the other by 5.

Test on Unit Three. Maxima and Minima

TEST I (40 minutes)

1. Write the derivative of the following:

   a. $3x^2 + 4x - 5 = 0$

   b. $\frac{x^2}{5} - \frac{2x}{3} + 7 = 0$

   c. $\frac{2x^2}{7} + x - \frac{1}{8} = 0$

2. Solve graphically $y = x^3 - 3x^2 - 9x + 5$ between the values $x = -2$ and $x = 4$, and from the graph determine a maximum and a minimum value of the curve.

3. Find the maximum or minimum value of:

   a. $2x^2 - x + 4 = y$

   b. $x^2 + 6x - 7 = y$

   c. $1 + 4x - x^2 = y$

4. By the use of the first and second derivatives find the maximum and minimum values of:

   a. $x^3 - 3x + 2 = y$

   b. $2x^3 - 11x^2 + 12x \quad 10 = y$

Test on Unit Four. Simultaneous Quadratic Equations

TEST I (40 minutes)

1. Solve by the graphic method and check your results by another method of solution

   a. $$\begin{aligned} x^2 + y^2 &= 52 \\ x - y &= 2 \end{aligned}$$

   b. $$\begin{aligned} x^2 - 3y^2 &= 7 \\ xy &= 2 \end{aligned}$$

2. Solve by any method

$$\begin{aligned} x^3 - y^3 &= 56 \\ x^2 + xy + y^2 &= 28 \end{aligned}$$

TEST II

1. Tell by inspection the curve that each of the following equations will produce

   a. $x^2 + y^2 = 49$

   b. $y^2 = 4x + 4$

   c. $x^2 + 6y^2 = 9$

d. $x^2 - 3y^2 = 2$

e. $xy = 10$

2. Solve graphically and check by another method of solution

$$x^2 + 4y^2 = 17$$
$$xy = 2$$

3. Solve and group roots correctly

$$x^2 + y^2 = 100$$
$$3x + 4y = 50$$

Test on Unit Five. Ratio, Proportion, and Variation

TEST I (40 minutes)

1. If the number of men required to do a piece of work varies directly as the size of the job and inversely as the number of days, write an equation connecting men (m) with job (w) and days (d).
2. The illumination of an object varies inversely as the square of its distance from a light, and directly as the candlepower of the light. If the illumination is just right for reading at a distance of 6′ from a 16 candlepower light, at what distance from a 25 candlepower light should a book be held? Show all work.
3. Given s equals the sum of three quantities that vary as x, x^2 and x^3 respectively. If $x = 1$, $s = 3$; if $x = 2$; $s = 6$; and if $x = 4$, $s = 16$. Express the value of s in terms of x.
4. Find each side of a triangle whose perimeter is n inches, the ratio of the sides being $a : b : c$.
5. If $a : b = b : c$ prove that

a. $a + b : b + c = \sqrt{a^2 - b^2} : \sqrt{b^2 - c^2}$

b. $a^2 - b^2 : a^2 + b^2 = a - c : a + c$

c. $ab - b^2 : bc - c^2 = b^2 : c^2$

d. $a^2 (a + b) : c(b + c) = a^2(a - b) : c(b - c)$

Test on Unit Six. Logarithms

TEST I (40 minutes)

1.

a. Define logarithm.

b. If 2.8659 is the log of a number, its colog is..........

c. Define antilog.

d. Name all the mathematical operations possible with logs.

2. Find

a. log .002547

  b. log 12340
  c. log .41282
  d. log 12347
  e. log .002

3. Find antilogs of
  a. $\bar{3}.47712$
  b. 2.66978
  c. .82575
  d. 9.89237
  e. .63994

4.
  a. Multiply by logs 240.8 x 46.09
  b. Divide by logs 43,826 x .72
  c. Simplify by logs $(.37)^4$
  d. Simplify by logs $.0175^{2/3}$

5. Simplify by logs $\sqrt[4]{\frac{42.2 \text{ x } .075}{3.64 \text{ x } .009}}$

6. What value of x satisfies the equation $A^{\frac{1}{x^2 + 2x + 4}} = \sqrt[3]{A}$

TEST II (40 minutes)

1. Logarithms are exponents. Yes or No.
2. Logarithms to the base................are called common logs
3. The log of 1 =
4. The Log 10 =
5. The integral part of a logarithm is called the............
6. The decimal part of a logarithm is called the............
7. The integral part of the logarithm of a number greater than ....than the number of significant figures to the left of the decimal point.
8. Rewrite this question in another form. Find the antilogarithm of 1.3927
9. The logarithm of $\frac{1}{\text{number}}$ is called ....
10. What is the logarithm of 8 to the base 2?
11. Find by logs
  a. (1.74) (.38) (24.6) =

b. $(7.146) \div (.6190) =$

c. $(1.04)^{10} =$

d. $\sqrt[4]{.0359} =$

e. $\sqrt[3]{.173} \times (1.04)^9 =$

f. $1.892 \times 10^{-3} \times 18.46 =$

12. The volume of a cube is 2150 cu. ins. Find an edge.

13. The volume of a cylinder is πr^2h. Find r when $V = 216$, $\pi = \frac{22}{7}$ and $H = 31.62$

14. Solve for x if $3^x = 60$

Test on Unit Seven. Arithmetic Progression

TEST I (40 minutes)

1. Find the arithmetic mean between $\sqrt{2}$ and $\sqrt{18}$.
2. What term of the progression 3, 6, 12 — — — — is 384?
3. What term of the progression 5, 10, 15 is 160?
4. A man earns $360 during his first year of work, and is given an increase of $50 a year for each succeeding year. What is his salary during the 10th year; and how much has he earned during the 10 years?
5. Derive the formula for the sum of n terms in A. P.
6. What is the tenth term of the series 5, —10, 20?
7. Insert 11 arithmetic means between —20 and 20.
8. What is the sum of the even numbers from 2 to 200 inclusive?
9. If the 3rd term of an A. P. is 15, and the 7th term is 37, what is the first term?

Geometric Progression

TEST II (40 minutes)

1. Write four numbers in G. P. in which $a = \sqrt{2}$ and $r = -\sqrt{3}$
2. State the common ratio in

   r, r^2, r^3 — — — —; 10, 5, $\frac{5}{2}$ — — — —; 2, $2\sqrt{2}$,

   4 — — — —; 2, $\sqrt{2}$, 1 — — — —; f, $\frac{g}{f}$, $\frac{g^2}{f^2}$. — —

3. Insert six geometric means between $\frac{27}{8}$ and $\frac{16}{81}$.

4. If the value of a motor car depreciates each year 40% of its value the preceding year, what is the value of the car at the end of 3 years if the initial cost of the car was $1800? (Consult Automobile Blue Book).

5. If a rubber ball is dropped from 4 ft. and rebounds ⅓ of its distance each time, over what distance will it have passed after 10 descents? Over what distance if it continues rebounding indefinitely?

6. The third term of a G. P. is $\frac{1}{12}$ and the eighth term is $\frac{8}{3}$, find the first two terms.

Test on Unit Eight. Binomial Theorem

Test I (40 minutes)

1. Expand $(3m + 4r)^3$ and simplify.
2. Expand and simplify $\left\{\frac{x}{a} - a^3\right\}^4$
3. Write first three terms of $(a - b)^{3/2}$ simplified.
4. Write first three terms of $(2 + b)^{-3}$ simplified.
5. Write first four terms of $(x^{-1/2} - 3x^2)^{10}$ simplified.
6. Expand, simplify, and leave with positive exponents

$$(a^{-1}x^{-2} - ax^{3/2})^3$$

7. Write the middle term of $(x^3 - y^2)^{16}$ simplified.
8. Find compound interest on $750.00 at 5% compounded annually for 3 years.

Test on Optional Unit. Numerical Trigonometry

Test I (40 minutes)

1. From the tables find
   a. sin 27°36′
   b. cos 37°42′
   c. tan 50°
   d. sin 72°
   e. cos 15°
2. Find the angle if
   a. $\sin x = .2756$
   b. $\cos x = .4540$
   c. $\tan x = 11.4301$
   d. $\tan x = .1405$
   e. $\cos x = .1908$

3. Given a right triangle with the following known

   a. b = 210, A = 32.5. Find c, a, B and area.

   b. A = 30, a = 12. Find c, b, B.

4. If a hillside has a slope of 5° how far up the hillside will a dam 10′ high force the water?

5. An automobile is going up a road which rises 150 ft. in every 1000 ft. of the length of the road. At what angle with the horizontal is the road rising?

Solid Geometry

Guiding Principles

In addition to the guiding principles for Plane Geometry, the following suggestions are made:

1. Proceed slowly at first, because many students have difficulty in making the transition from Plane Geometry to Solid Geometry.

2. Before a student attempts to draw a figure, he must understand the relative positions of the lines and planes involved. For many theorems it is possible to make simple figures either by holding pencils and pieces of cardboard or by fastening together sticks, string, and cardboard.

3. A sphere is defined as a surface. Similar usages are recommended for polyhedron, cylinder, and cone. Words such as dodecahedron and icosahedron may be replaced by equivalent phrases. The English plurals polyhedrons, and frustums are recommended. "Diagonal" is used for polygons and also for polyhedrons.

4. Emphasize the part which the formulae of Plane Geometry play in Solid Geometry; for example, in finding the lateral areas and total areas of prisms, pyramids, and frustums.

5. Give many exercises that require thoughtful applications of facts learned.

Introductory Unit

1. Some of the more obvious contrasts between Plane Geometry and Solid Geometry may furnish a suitable beginning. For example:

Plane Geometry	*Solid Geometry*
a. Points, straight lines, line segments, circles, and arcs are used.	a. Points, lines, circles, prisms, pyramids, cones, cylinders, and spheres are used.
b. Everything is restricted to one plane.	b. There is no restriction to one plane.
c. A drawing is made either congruent or similar to the original figure. For example, the drawing of a square is always a square.	c. It is necessary to use "perspective." For example, a cube has six squares, but the drawing of a cube does not contain six squares.
d. Two lines are either parallel or intersecting.	d. Two lines may be neither parallel nor intersecting.

2. By holding pencils in various positions the student will learn that some theorems of Plane Geometry are true in Solid Geometry and others are not true. For example, the statement "Two lines parallel to a third line are parallel to each other" is true in Solid Geometry;

but the statement "Two lines perpendicular to a third line are parallel to each other" is not true unless the three lines are in one plane.

3. The assumptions of Solid Geometry include (a) those of Plane Geometry which are valid in Solid Geometry and (b) additional assumptions, such as, "If two points of a straight line are in a plane, the line is wholly in the plane," and "One plane and only one can be passed to contain a straight line and a point without."

4. If an informal proof of Fundamental Theorem 1 is given in the Introduction, a more satisfactory start can be made in Unit Two, because Fundamental Theoren 2 is better adapted for this purpose.

FUNDAMENTAL THEOREMS

1. If two planes meet, they intersect in a straight line.

2. If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane of the two lines.

3. Every perpendicular to a given line at a given point lies in a plane perpendicular to the given line at the given point.

4. Through a given point (internal or external) there can pass one and only one line perpendicular to a plane.

5. Two lines perpendicular to the same plane are parallel.

6. If two lines are parallel, every plane containing one of the lines and only one is parallel to the other.

7. Two planes perpendicular to the same line are parallel.

8. If two parallel planes are cut by a third plane, the lines of intersection are parallel.

9. If two angles not in the same plane have their sides respectively parallel in the same sense, they are equal and their planes are parallel.

10. If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.

11. If a line is perpendicular to a given plane, every plane which contains this line is perpendicular to the given plane.

12. If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.

13. The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.

14. An oblique prism is equal in volume to a right prism whose base is equal in area to a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism.

15. The opposite faces of a parallelepiped are congruent.

16. The plane passed through two diagonally opposite edges of a parallelepiped divides the parallelepiped into two triangular prisms which are equal in volume.

17. The lateral area of a prism (a circular cylinder) is equal to the product of a lateral edge (element) by the perimeter (circumference) of a right section.

18. The volume of a prism (including any parallelepiped) or of a circular cylinder is equal to the product of its base by its altitude.

19. If a pyramid (or a cone) is cut by a plane parallel to the base:

a. The lateral edges (or elements) and the altitude are divided proportionally.

b. The section is a figure similar to the base.

c. The area of the section is to the area of the base as the square of the distance from the vertex is to the square of the altitude of the pyramid (or cone).

20. Two triangular pyramids having equal altitudes and bases equal in area have equal volumes.

21. The lateral area of a regular pyramid or a right circular cone is equal to half the product of its slant height by the perimeter (circumference) of its base.

22. The volume of a pyramid or a cone is equal to one-third the product of its base by its altitude.

23. All points on a circle of a sphere are equidistant from either pole of the circle.

24. On any sphere a point which is at a quadrant's distance from each of two other points not the extremities of a diameter is a pole of the great circle passing through these two points.

25. If a plane is perpendicular to a radius at its extremity on a sphere, it is tangent to the sphere.

26. A sphere can be inscribed in or circumscribed about any tetrahedron.

27. If one spherical triangle is the polar of another, then reciprocally the second is the polar triangle of the first.

28. In two polar triangles each angle of either is the supplement of the opposite side of the other.

29. Two symmetric spherical triangles have equal areas.

30. The area of a sphere is equal to the area of four great circles.

$$A = 4 \pi r^2.$$

31. The area of a spherical polygon is equal to the area of a lune whose angle is half the spherical excess of the spherical polygon.

32. The volume of a sphere is equal to one-third the product of its radius by its area.

$$V = 4/3 \pi r^3.$$

Subsidiary Theorems

1. If from an external point a perpendicular and obliques are drawn to a plane, (a) the perpendicular is shorter than any oblique; (b) obliques meeting the plane at equal distances from the foot of the perpendicular are equal; (c) of two obliques meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the longer.

2. If two lines are cut by three parallel planes, their corresponding segments are proportional.

3. Between two lines not in the same plane there is one common perpendicular, and only one.

4. The sum of any two face angles of a trihedral angle is greater than the third face angle.

5. The sum of the face angles of any convex polyhedral angle is less than four right angles.

6. If three face angles of one trihedral angle are equal respectively to the three face angles of another, the trihedral angles are either congruent or symmetric.

7. a. The locus of points equidistant from two given points is the plane perpendicular to the line segment joining them, at its midpoint. b. The locus of points equidistant from two given intersecting planes is the pair of planes bisecting the dihedral angles formed by the given planes.

8. The bases of a cylinder are congruent.

9. The lateral area of a frustum of a regular pyramid or a cone of revolution is equal to half the product of the slant height by the sum of the perimeters (circumferences) of the bases.

10. The volume of a frustum of a pyramid or a cone is equal to one-third the product of its altitude by the sum of the lower base, the upper base, and the mean proportional between the bases.

11. The volumes of two tetrahedrons that have a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the three edges of these trihedral angles.

12. In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.

13. Two similar polyhedrons can be separated into the same number of tetrahedrons similar each to each and similarly placed.

14. The volumes of two similar tetrahedrons are to each other as the cubes of any two corresponding edges.

15. The volumes of two similar polyhedrons are to each other as the cubes of any two corresponding edges.

16. There cannot be more than five regular polyhedrons.

17. The volume of a prismatoid is equal to one-sixth the product of its altitude by the sum of its upper base, its lower base, and four times its mid-section. (without formal proof)

18. If a plane intersects a sphere, the intersection is a circle.

19. Two spherical triangles on the same sphere are either congruent or symmetric if (a) two sides and the included angle of one are equal to the corresponding parts of the other; (b) two angles and the included side of one are equal to the corresponding parts of the other; (c) the three sides of the one are equal respectively to the three sides of the other; (d) the three angles of the one are equal respectively to the three angles of the other.

20. Each side of a spherical triangle is less than the sum of the two other sides.

21. The sum of the sides of a spherical polygon is less than 360°.

22. The sum of the angles of a spherical triangle is greater than 180° and less than 540°.

SIX TEACHING UNITS

Unit One. Introduction. (Suggested time—two weeks)

Relation of Plane Geometry to Solid Geometry. Assumptions. Figures and drawings. Fundamental Theorem 1. This unit has already been given.

Unit Two. Lines and Planes (Four weeks)

Fundamental Theorems 2-12.
Subsidiary Theorems 1-7.

Unit Three. Prisms and Cylinders (Two weeks)

Fundamental Theorems 13-18.
Subsidiary Theorems 8.

Unit Four. Pyramids and Cones (Two weeks)

Fundamental Theorems 19-22.
Subsidiary Theorems 9 and 10.

Unit Five. Tetrahedrons and Other Polyhedrons (One week)

Subsidiary Theorems 11-17.

Review two weeks.

Unit Six. Spheres (Three weeks)

Fundamental Theorems 23-32.
Subsidiary Theorems 18-22.

General review two weeks.

The time suggested for these six units is eighteen weeks. If more time is available, it can be spent on Cavalieri's theorem, or on further study of similar solids, or spheres.

UNIT TESTS—SOLID GEOMETRY*

Test on Unit One

1. a. Is every triangle a plane figure? Why? b. Is every quadrilateral a plane figure? Why?

2. Through a given point how many lines can be drawn perpendicular to a given line, (a) if the point is on the line, (b) if the point is not on the line? Why?

3. If two parallel lines are cut by a transversal, are the three lines in one plane? Why?

4. How many planes can be passed to contain (a) a given line, (b) two intersecting lines, (c) two parallel lines?

Test on Unit Two

1. Tell how to construct a line perpendicular to a given plane at a given point in the plane. Why is this construction correct?

2. a. If two planes are perpendicular to the same line, are they parallel? b. If two planes are perpendicular to the same plane, are they parallel? c. If two lines are perpendicular to the same line, are they parallel? d. If two lines are perpendicular to the same plane, are they parallel? e. If two planes are parallel to the same line, are they parallel to each other?

3. Prove:—If two intersecting planes are each perpendicular to a third plane, their line of intersection is also perpendicular to that plane.

Test on Unit Three

1. Prove that the lateral area of a prism is equal to the product of the perimeter of a right section and a lateral edge.

2. The base of a right prism is inclosed by a square 4 in. on a side and its altitude is 10 in. Find its volume. Find the lateral area and the volume of the right circular cylinder whose upper and lower bases are circumscribed about those of this prism.

3. The three edges at a vertex of a rectangular solid (rectangular parallelepiped) are 1 in., 2 in., and 3 in. Find the diagonals of the three faces at that vertex and a diagonal of the solid.

4. A base of an oblique prism is inclosed by a regular hexagon 2 in. on a side. A lateral edge is 8 in. and makes an angle of 60° with a perpendicular from its upper end to the plane of the lower base. Its volume is . . .

Test on Unit Four

1. If a pyramid is cut by a plane parallel to the base, the area of the section is to the area of the base as the square of the distance from the vertex is to the square of the altitude of the pyramid. Prove.

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*Due to the large amount of space required to include types of tests more objective in character, these have been omitted. Teachers are urged to develop these type tests and to use them most frequently.

2. Find the lateral area and the volume of a regular pyramid whose lateral edge is 13 in. and whose base is a square 10 in. on a side.

3. Find the volume and the total area of a right circular cone whose slant height is 6.5 in. and whose altitude is 6 in.

Test on Unit Five

1. A pyramid whose altitude is 12 in. weighs 256 pounds. Find the weight of the pyramid cut off by a plane parallel to the base, bisecting the altitude.

2. Corresponding edges of two similar polyhedrons are 9 in. and 13.5 in. The total area of the first polyhedron is 216 sq. in. and its volume is 162 cu. in. Find the total area and volume of the second polyhedron.

3. There are three regular polyhedrons whose faces are inclosed by equilateral triangles. How many faces, how many edges, and how many vertices has each of these regular polyhedrons?

Test on Unit Six

1. Prove:—In two polar triangles each angle of either is the supplement of the opposite side of the other.

2. Find the area and the volume of a sphere whose radius is 5 in.

3. What is the area of a spherical triangle whose angles are 110°, 90°, and 80°, if the radius of the sphere is 6 in.?

General Test on Solid Geometry

This examination consists of three parts, which the teacher may find it convenient to give on three different days. Parts I and II are based on Fundamental Theorems and Constructions, Part III involves the Subsidiary Theorems also. If a student takes all three parts, Part I receives 30%, Part II 40%, and Part III 30%. The student who has not studied the Subsidiary Theorems will not take Part III and will receive 50% on Part I and 50% on Part II.

Part I

Answer these questions but do not write out proofs:

1. The radius of the base of a right circular cylinder equals the radius of a sphere and its altitude equals the diameter of the sphere. Find the ratio of the area of the sphere to the total area of the cylinder. Find the ratio of their volumes.

2. The volume of a circular cone is 9π cu. in. and its altitude is 12 in. Find the diameter of its base.

3. The altitude of a regular pyramid is 18 in. Its base is a square 6 in. on a side. Find the volumes of the three parts into which the pyramid is divided by two planes which are parallel to the base and divide the altitude into three equal parts.

4. The three sides of a triangle are 3 in., 4 in., and 5 in. Find the total area of the figure obtained by revolving the triangle about its longest side.

5. Find the volume of a regular pyramid which has six edges each 6 in.

Part II

Prove these theorems:

1. If two angles not in the same plane have their sides respectively parallel in the same sense, they are equal and their planes are parallel.

2. If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.

3. If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane of the two lines.

4. Two symmetric spherical triangles have equal areas.

Part III

Answer these questions, but do not write out a proof unless you are asked to do so:

1. A given point, A, is in a given plane. What is the locus of points which are 5 in. from A and also 2 in. from the given plane?

2. Prove that the sum of the angles of a spherical triangle is greater than 180° and less than 540°.

3. An edge of a regular octahedron (a regular polyhedron of eight faces) is 2 in. Find its volume.

4. A lateral edge of a frustum of a regular pyramid is 5 in. The bases are squares 12 in. and 6 in. on a side. Find the total area and the volume.

5. A light illuminates one-fourth of a sphere. How far is the light from the center of the sphere?

Plane Trigonometry

Unit I. Trigonometric Functions of Acute Angles (Three weeks)

I. Angular Measure

A. This should be taught by both sexagesimal and circular measure.

II. Trigonometric Functions

A. At this time link up new ideas with the student's past experience in Plane Geometry. Show by the use of a number of right triangles with a common acute angle that these ratios remain unchanged so long as the angle remains unchanged, but they change as the angle changes.

B. On the squared paper have the students make a table of tangents. With measurements taken accurately, the student will be surprised to find his results comparing favorably to at least three figures with the table of natural functions. Similar studies may be made of sines and cosines.

III. Representation of Functions by Lines

A. This is taught by the use of the unit circle. All ratios, or functions, are so set up that the denominator is the radius.

IV. Changes in Functions as the Angle Changes

A. This is taught by graphical representation through numerous examples of the increase and decrease of functional line values. It is recommended that different colored crayon be used to designate the various functions. Students should be drilled to harmonize angular size with line values.

V. Significance of Cofunctions

A. This is taught by the use of the right triangle. Students should be encouraged to experiment with numerous right triangles on squared paper, and thus to arrive at correct conclusions in regard to the equivalence of cofunctions.

VI. Inter-relationship of Functions of all Angles

A. This is best shown and taught by the use of the table (Wentworth Smith's) shown as follows,—

$$\begin{array}{ccc} & \tan x & \\ \sin x & & \sec x \\ \cos x & & \csc x \\ & \cot x & \end{array}$$

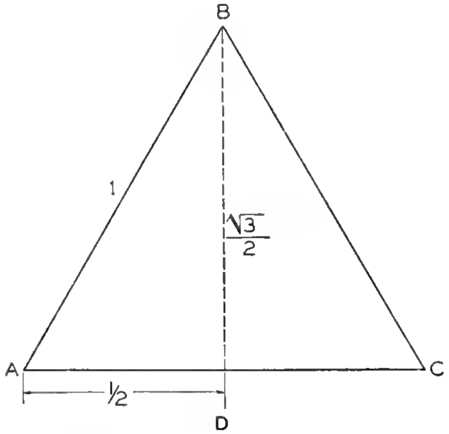
"In the above diagram, any function is equal to the product of the two adjacent functions, or to the quotient of either adjacent function divided by the one beyond it."

VII. Computation of the Other Five Functions if One is Given

A. This is best developed and taught by the method suggested in the text used.

VIII. Functions of 30°, 45°, 60°

A. A suggested class room procedure for the development of these angles follows,—



1. How could we draw a triangle containing a 60° angle.
2. When some student suggests how to draw the triangle sketch the figure on the board.
3. What do you know about the sides of triangle ABC?
4. How many degrees in each angle of triangle ABC?
5. Let us see if we can represent the functions of ABC?
6. Before you can write the functions of angle A, angle A must be an angle of what kind of a triangle?
7. Then draw the perpendicular BD at the suggestion of a pupil.
8. In triangle ABC, what is the relation of AD to AB?
9. When the pupils see that AB is twice AD, place the numerical values for AD and AB on the figure, and then determine the value of BD.
10. Sin angle A or sin 60° = ?

    Cos angle A or cos 60° = ?

    Tan angle A or tan 60° = ?
11. How many degrees in angle ABD?
12. Sin angle ABD or sin 30° = ?

    Cos angle ABD or cos 30° = ?

    Tan angle ABD or tan 30° = ?
13. In a similar way develop the functions of 45°.

B. After the values of these functions have been developed, the student should have well fixed in mind these values. An aid in remembering these values follows,—

	0°	30°	45°	60°	90°
Sin	$\frac{1}{2}\sqrt{0}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{4}$
Cos.	$\frac{1}{2}\sqrt{4}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{0}$

Unit II. Solution of Triangles by use of Natural Functions (Three weeks)

I. RIGHT TRIANGLES

A. Acute angle and hypotenuse given

1. The use of problems of this type should be introduced by the use of two very simple problems. The first problem should be to find the side opposite the given acute angle, and the second problem should be to find the sine adjacent to the given acute angle. Facility in the selection of the proper function for use will come through practice.

B. Acute angle and opposite leg given.

1. Problems used here should include

   a. The use of the Sin.

   b. The use of the Cos.

   Use numerous simple problems to drive home the use of these functions.

C. Acute angle and adjacent leg given.

1. The suggestions given in (A) and (B) apply equally well here.

D. Two legs given.

1. In this type of solution, use simple problems involving the use of angles of elevation and depression. Instruct the student that both can be defined by the same definition, to wit., "the angle between the line of sight and the horizontal."

E. Areas.

1. Problems to determine areas in this unit should be limited to the following cases:

   a. Given two sides about the right angle.

   b. Given the hypotenuse and one leg.

   c. Given the hypotenuse and an acute angle.

2. In cases (a) and (b) above, use the principles of Geometry. In case (c) it is of course necessary to use the functions of the given acute angle to determine the two legs.

3. The formulae for the areas of right triangles should also be derived in this unit.

F. The isosceles triangle.

1. The development of the solution of one problem of this type should be sufficient for the pupils to fully understand the application of the principles of the right triangle to the solution of isosceles triangle.

G. The regular polygon.

1. If care is taken in the selection of problems of this type at first, the students should experience no new difficulty.

2. An interesting tie-up with Geometry is to check with the proposition, "The sum of the interior angles of a polygon is equal to (N-2) st. angles."

II. OBLIQUE TRIANGLES (optional presentation)

A. Reduction formulae for sin and cos for the second quadrant.

1. The development of the formulae for Sin (180-A) and Cos (180-A) should precede the development of formulae for Sin (90 + A) and Cos (90 + A).

B. Solution when two angles and a side are given by dividing the triangle into two right triangles.

C. Law of Sines.

D. Solution when two angles and a side are given by using Law of Sines.

1. It is suggested that the same problem be used as in (B) above.

E. Solution when the three sides are given by drawing a perpendicular to one side and determining one of the segments formed and then by use of the cos function, find the angles.

F. Law of Cosines.

G. Solution when the three sides are given by using Law of Cosines.

1. It is suggested that the same problems should be used here as in (E) above.

2. Although this presentation is optional with the teacher, it has been found by trial that the approach to oblique triangles in this manner is advisable if extreme care is used in the selecting of material.

Unit III. Trigonometric Functions of Any Angle (Five weeks)

I. EXPANSION OF THE IDEA OF LINE FUNCTIONS IN UNIT CIRCLES

If a circle of unit radius is described about the vertex of an angle, the line definitions are:

A. The sine of the angle is represented by the perpendicular upon the initial line from the intersection of the terminal line with the circumference.

B. The cosine of the angle is represented by the segment of the initial extending from the vertex to the sine.

C. The tangent of the angle is represented by a line tangent to the circle at the beginning of the first quadrant and extending from the point of tangency to the terminal line.

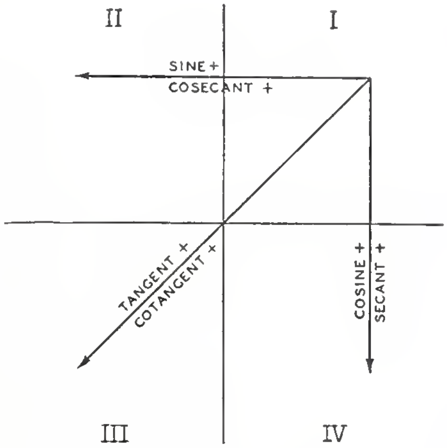
D. The cotangent of the angle is represented by a line tangent to the circle at the beginning of the second quadrant, and extending from the point of tangency to the terminal line.

E. The secant of the angle is represented by the segment of the terminal line extending from the vertex to the tangent.

F. The cosecant of the angle is represented by the segment of the terminal line extending from the vertex to the cotangent.

II. Positive and Negative Quantities

The relative positions of the line functions upon the X and Y axes determine the positive or negative values of the functions. Excellent use can be made of the quadrant table.



III. Coordinates of a Point in a Plane

By placing the vertex of the angle at the origin, and the initial line upon the positive X axis, the coordinates become the same as those used in Algebra.

IV. Angles of Any Magnitude (any number of turns)

Same as an angle less n times 360°.

V. Changes in the Functions as Angles Increase from 0° to 360°, with a discussion of the turning points and their significance (unit circle)

A. Clockwise—negative angle.

Counterclockwise—positive angle.

B. Variation in values.

VI. Functions of Negative Angles

A. Any function of the angle (—X) is equal in numerical value to the same function of X. Its sign, however, depends on the direction of the line representing it. The functions of the angle (—X) are the same as for those of (360°—X).

VII. Further Development of the Reduction of Angles to the First Quadrant.

A. A function of an acute angle is equal to the cofunction of its complementary angle.

B. The sine of an angle is equal to the sine of its supplement, and the cosine, tangent and cotangent are each equal to minus the same functions of its supplement.

VIII. FUNCTIONS OF ANGLES THAT DIFFER BY 90°

A. Any function of each of the angles (180°—X), (180°+X), (360°—X) is equal in numerical value to the same function of X. Its sign, however, depends on the direction of the line representing it.

B. Any function of each of the angles (90°—Y), (90°+Y), (270°—Y), (270° + Y), is equal in numerical value to the cofunction of Y. Its sign, however, depends on the direction of the line representing it.

IX. FUNCTIONS OF THE SUM OF TWO ANGLES

Derivation of formulae should be stressed, followed by application to problems.

X. FUNCTIONS OF THE DIFFERENCE OF TWO ANGLES

Derivation of formulae should be stressed, followed by application to problems.

XI. FUNCTIONS OF DOUBLE AN ANGLE

Show the substitution of two equal quantities in formulae for the sum of two angles, and the result.

XII. FUNCTION OF HALF AN ANGLE

Show the substitution of ½X in formulae in XI.

XIII. SUMS AND DIFFERENCES OF TRIGONOMETRIC RATIOS (optional)

Let $U = (X+Y)$ and $V = (X—Y)$, then $X = \frac{1}{2}(U+V)$, and $Y = \frac{1}{2}(U—V)$, and by substitution in formulae for sums and differences of trigonometric functions, the ratios are secured.

XIV. INVERSE TRIGONOMETRIC RATIOS (functions)

The expressions $\sin^{-1} a$, $\cos^{-1} a$, $\tan^{-1} a$, etc., denote respectively an angle whose sin is a, an angle whose cosine is a, an angle whose tangent is a, etc. They are called the inverse sine of a, the inverse cosine of a, the inverse tangent of a, etc., and are the inverse trigonometric functions. $\text{Sin}^{-1} a$ is an angle whose sine is equal to a, and hence denotes, not a single definite angle, but each and every angle whose sine is a. The inverse functions are sometimes called anti-trigonometric functions.

XV. GRAPHICAL REPRESENTATION

All functions contain two variables, and for every value of the angle X there is a corresponding value of Y. It is possible, therefore, to represent all the values of any trigonometric function by a curve, and the character of this curve will reflect the behavior of the function as the angle varies.

XVI. RADIAN MEASURE (or π measure)

A. An arc whose length is equal to the radius will subtend the angle

$(180/\pi)^\circ$. This angle is the unit of circular measure and is called a radian. Now explain the use of $\pi/2$, π, etc., denoting $\pi/2$ radians, π radians, etc.

B. To reduce a certain number of radians to degrees multiply by $180/\pi$.

C. To reduce a certain number of degrees to radians multiply by $\pi/180$.

Unit IV. Fundamental Identities and Trigonometric Equations of Simpler Form (One week)

(In addition, for college preparation, it is recommended that one or two identities be included hereafter in daily assignments until sufficient mastery has been attained.)

I. IN PROVIDING IDENTITIES, the teacher should cause the beginning student to see that there can be vertical equality as well as horizontal equality; in other words he should teach the student to make substitutions directly below their equals. An example of vertical equality is shown below:

$$\begin{array}{llcl}
\text{Sin } 3X = & & & \\
\text{Sin }(2X+X) = & \text{Sin } 2X \quad \cdot \text{Cos}X & + & \text{Cos } 2X \quad \cdot \text{Sin}X \\
& \qquad \| & & \qquad \| \\
& = 2\,\text{Sin}X\ \text{Cos}X \cdot \text{Cos}X & + & (\text{Cos}^2X - \text{Sin}^2X) \cdot \text{Sin}X \\
& = 2\,\text{Sin}X\ \text{Cos}^2X & + & \text{Cos}^2X\ \ \text{Sin}X - \text{Sin}^3X \\
& \qquad\quad \| & & \qquad \| \\
& = 2\,\text{Sin}X\ (1 - \text{Sin}^2X) & + & (1 - \text{Sin}^2X)\ \text{Sin}X - \text{Sin}^3X \\
& = 2\,\text{Sin}X - 2\text{Sin}^3X & + & \text{Sin}X - \text{Sin}^3X - \text{Sin}^3X \\
& = 3\,\text{Sin}X - 4\text{Sin}^3X & &
\end{array}$$

After a student has had a certain amount of practice in proving identities, he may feel fairly safe in eliminating some of the detailed steps, and arrive at his solution much more quickly and without sacrificing any of the essentials. However, the safest plan is to adhere to strict sequence.

II. IN SOLVING EQUATIONS, the teacher should cause the student *first* to decide which function will give the simplest and most direct solution; *second,* to make the substitution vertical as suggested above in identities; and *third,* to remember that the solution is not complete until all values are found in accordance with the conditions of the problem.

Unit V. Logarithms (Two weeks)

I. HISTORY, DEFINITION, AND IMPORTANCE

A. A brief History of Logarithms is contained in most textbooks. For extended study the student should consult a good library where a complete History of Mathematics can be found.

B. In defining logarithms, the student should early be made to see that logarithms are only exponents, and that the principles of exponents as learned in Algebra apply equally well to logarithms.

C. Through a succession of simple to difficult problems, the student should be made to see the practical value of logarithms and the impossibility of the solution of certain types of problems without their use.

II. ELEMENTARY THEORY AND COMPUTATIONS

A. The teaching of Elementary Theory and Computations begins with the student learning the two parts of each logarithm, and how these parts are obtained. The rules for finding the characteristic should be stressed, and sufficient practice given to the student until he has completely mastered this fundamental. He should also be taught to rely not altogether on his rules, but to use common sense as well. After the student has learned without fail to find the characteristic, he might be shown this simple mnemonic; having given the number, he should think of an imaginary mark directly to the right of the first significant figure, and then count right or left to the decimal point. If he counts right, the characteristic is positive; if to the left, the characteristic is negative. For example: having given the number 24638.45, first put a mark to the right of 2. Thus 2′4638.45. Second, count right to the decimal point. The characteristic is 4. Again, having given the number .074638. First, put a mark to the right of 7. Thus .07′4638. Second, count left to the decimal point. The characteristic therefore is 2. The use of this simple mnemonic brings into play the elementary idea of positive and negative according as you count right or left.

III. LOGARITHMS OF PRODUCTS, QUOTIENTS, POWERS, ROOTS

A. The teaching of the logarithms of products, quotients, roots, powers, should be preceded by a short review of the Laws of Exponents as previously learned in Algebra. The use of a few samples will suffice to convince the student of the extreme usefulness of logarithms in mathematical endeavor.

IV. ALGEBRAIC PROBLEMS INVOLVING THE DIFFERENT PROCESSES. EXPONENTIAL EQUATIONS

A. Problems involving these different processes should be selected from the textbook used. A sufficient number should be assigned to satisfy the teacher of the proper achievement of his pupils.

Unit VI. Solution of Triangles by use of Logarithms (Four weeks)

I. RIGHT TRIANGLES

II. OBLIQUE TRIANGLES

A. Given one side and two angles.

B. Given two sides and an angle opposite one of them.

C. Law of Tangents.

D. Given two sides and the included angle.

E. The half angle formulas.

F. Given three sides.

G. Areas.

H. Radii of Inscribed and Circumscribed Circles.

The development of this unit should be left largely with the pupil. Here he must put to use the theories learned in preceding units. His own resourcefulness and initiative must be put to the test. He must learn how to attack a problem intelligently, what methods to use for a correct solution, and how to carry on the work to a successful fruition. There is no short cut to success here.

Problems for the test on this unit should be selected by the instructor from the textbook used, and from supplementary sources. The particular type of material contained herein does not lend itself so readily to new type tests.

Unit VII. Supplementary Topics (Two weeks, if twenty week course)

I· SLIDE RULE

II. HISTORICAL SKETCH OF TRIGONOMETRY

III. COMPLEX NUMBERS

IV. TRIGONOMETRIC SERIES

V. EXPONENTIAL SERIES

VI. DE MOIVRES THEOREM

VII. FIELD TRIGONOMETRY

VIII. GRAPHIC PRESENTATION OF LOGS

IX. SEMI-LOGARITHMIC CHARTS

UNIT TESTS—PLANE TRIGONOMETRY

Test on Unit I

TEST A

Directions: If a statement is true, put a plus (+) sign on the short dotted line before it; if it is false, put a zero (0). Tables are not to be used. Use the margin for figuring. Time, 15 minutes.

......1. In a right triangle when the acute angles are 30° and 60°, the side opposite the 30° angle is half the side opposite the 60° angle.

......2. Each function of an acute angle is equal to the conamed function of its complementary angle.

......3. The Pythagorean theorem ($a^2 + b^2 = c^2$) is very little used in Trigonometry.

......4. The sine of twice an angle is equal to twice the sine of the angle.

......5. If Sin X = 4/5, Cos X = 1/5.

......6. The secant is the reciprocal of the cosecant.

......7. The angle of depression is defined as the angle between the line of sight and the horizontal.

......8. As the angle increases, the sine increases more rapidly than the tangent.

......9. If you bisect the angle, you also bisect the tangent.

.....10. It is possible to construct an angle with a tangent of 4/3.

.....11. $\text{Sec}^2 A + \text{Tan}^2 A = 1$.

.....12. The hypotenuse of an isosceles right triangle is equal to a side multiplied by $\sqrt{2}$.

.....13. The sine and cosine of a 45° angle are not equal in value.

.....14. Tan A Cos A = Sin A.

.....15. Sin(20° + 30°) = Sin 20° + Sin 30°.

.....16. The tangent of an angle is equal to the reciprocal of the tangent of the complement.

.....17. If the secant of an angle is 2, the csc is 1/2.

.....18. Sines and Cosines are never greater than 1.

.....19. In a unit circle, the secant of an angle is greater than the tangent.

.....20. As the angle increases from 0° to 90°, the cofunctions increase in value.

.....21. $\text{Cot}^2 A + 1 = \text{Csc}^2 A$.

.....22. If cos X = 3/5, then tan X = 4/3.

.....23. If the tangent of 30° is $1/3\sqrt{3}$, the cotangent of 60° is $\sqrt{3}$.

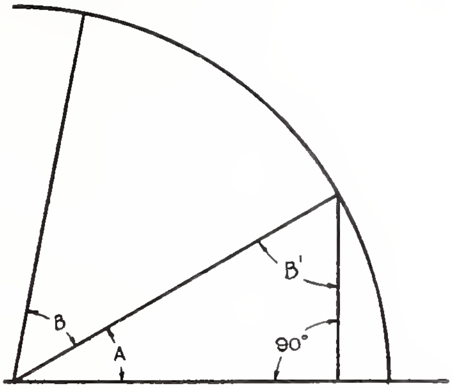
.....24. The functions of a 45° angle are never equal to one-half the corresponding functions of 90°.

.....25. In a unit circle, the sine of an angle is not equal to one-half the chord of twice the angle.

TEST B

Exercise to test student resourcefulness in meeting certain situations. Time, 30 minutes.

In the accompanying figure,—



1. Find the size of angle A in degrees and minutes.
Ans.______________________________

2. Find the size of angle B in degrees and minutes.
Ans.______________________________

3. Draw the csc of A and designate it.
Ans.______________________________

4. Draw the Tan of A and designate it.
Ans.______________________________

5. Draw the Sin (A + B) and designate it.
Ans.______________________________

6. What functions increase as the angle decreases?
Ans.______________________________

7. Draw a line which shall be equal in length to Cos A + Cos B.
Ans.______________________________

8. Does Cos A = Cos B?
Ans.______________________________

9. (a). If A were 30°, what would be the Cos of B′?
Ans.______________________________
(b). If A were 30°, what would be the Tan of B′?
Ans.______________________________

10. Which of the following angles could be constructed, supposing the values of the functions to be as follows,—

$\text{Sin } A = \frac{5}{4}$, $\text{Cos } A = \frac{1}{3}$, $\text{Tan } A = \frac{4}{3}$, $\text{Sec } A = 2$

Ans.______________________________

Test on Unit II

Directions: If a statement is true, put a plus (+) sign on the short dotted line before it; if it is false, put a zero (0). Tables are not to be used. Use the margin for figuring. Time, 25 minutes.

Given a right triangle ABC with C being the right angle.

......1. If c is 2 inches and a is 1 inch, then angle A is 60°.
......2. If A is 54°, then b/a is the tangent of 36°.
......3. If A is 54°, and c is 17 in., then $b = 17 \cos A$.
......4. If A is 54°, and c is 17 in., then a is 18.753 inches.
......5. $a = c\ \mathrm{Sin}\ A$
......6. $b = a\ \mathrm{Tan}\ A$
......7. $c = b\ \mathrm{Sin}\ B$
......8. $a^2 + c^2 = b^2$
......9. $\mathrm{Sin}\ C = 0$
.....10. If A is 45°, then Tan A is 1.4142.
.....11. If B is 37°23′, then Sin B is 1.6072.
.....12. If A is 48°37′, then Tan A is .7301.
.....13. If A is 73°42′, then Cot A is 1.3249.
.....14. If B is 14°54′, then Cos B is .0664.
.....15. If B is 60°, then Sin A is .5000.
.....16. If b is 8 inches, and c is 13 inches, the area is $4\sqrt{105}$.
.....17. $\mathrm{Sin}\ (180 - A) = \mathrm{Sin}\ A$
.....18. $\mathrm{Cos}\ (90 + A) = -\mathrm{Sin}\ A$
.....19. $\mathrm{Cos}\ (180 - A) = \mathrm{Cos}\ A$
.....20. $\mathrm{Sin}\ (90 + A) = \mathrm{Cos}\ A$
.....21. $\mathrm{Sin}\ 130° = .5000$.
.....22. $a : b = \mathrm{Sin}\ A : \mathrm{Sin}\ B$
.....23. $a^2 = b^2 + c^2 - 2\,bc\ \mathrm{Cos}\ A$
.....24. $\mathrm{Cos}\ B = \dfrac{a^2 + c^2 - b^2}{2\ ac}$
.....25. $\mathrm{Cos}\ C = \dfrac{a^2 + c^2 - b^2}{2\ ab}$

Test on Unit III

Directions: If a statement is true, put a plus (+) sign on the short dotted line before it; if it is false, put a zero (0). Tables are not to be used. Use the margin for figuring. Time, 30 minutes.

......1. The point (a, b) is always in quadrant I.
......2. $\mathrm{Tan}\ \pi = 0$.
......3. $\mathrm{Sin}\ 2A = 1 - 2\sin^2 A$.
......4. $\mathrm{Sin}\ (-A) = -\mathrm{Sin}\ A$
......5. $\mathrm{Cos}\ (A - B) = \mathrm{Cos}\ A \cos B - \mathrm{Sin}\ A\ \mathrm{Sin}\ B$
......6. $\mathrm{Sin}\ 150° = .5$.

......7. $36^\circ = \pi/5$ radians.

......8. The angle -400° is in the fourth quadrant.

......9. The hands of a clock generate a positive angle.

.....10. Cot 172° is less than 0.

.....11. $\text{Sin}\ \frac{A}{2} = \sqrt{\frac{1 + \cos A}{2}}$

.....12. The point $(-a, -b)$ is not in the third quadrant.

.....13. $\text{Tan}^{-1}\ 1 = 45^\circ$

.....14. $\text{Cos}\ (-A) = +\cos A$

.....15. $\text{Sin}\ (180 + x) = -\sin x$

.....16. $\text{Sin}\ 735^\circ = \text{Sin}\ 15^\circ$

.....17. An angle with a negative cosine and a positive tangent is in the second quadrant.

.....18. An angle with a secant of 2 is in the fourth quadrant.

.....19. $\text{Sin} - 137^\circ = - \text{Sin}\ 43^\circ$

.....20. As an angle decreases from 90° to 0°, the cofunctions decrease.

.....21. $\text{Sin}\ (A + B) = \text{Sin}\ A \cos B + \cos A \sin B$

.....22. $\text{Sin}\ (270 + x) = -\sin x$

.....23. As angle x increases from 0° to 90°, the sin increases more rapidly than the tangent.

.....24. A radian is a central angle subtended by a chord the length of a radius.

.....25. $\text{Sin}\ 240^\circ = 4 \sin 60^\circ$.

.....26. $\text{Sin}\ 150^\circ + \text{Tan}\ 135^\circ + \text{Cos}\ 120^\circ = -1$

.....27. The periodicity of Sin x is 360°

.....28. The value of a radian is $57^\circ 17' 45''$.

.....29. Any trigonometric function of $(90-A)^\circ$ is equal to the cofunction of A°.

.....30. The terminal line of angle -35° is the same as that of angle 325°.

.....31. The functional values of 1085° are the same as those of 5°.

.....32. $\text{Cos}\ (\tan^{-1}x) = \frac{1}{2}\sqrt{2}$ is 1

.....33. $\text{Sin}\ 68^\circ 15' = \cos 0^\circ 55'$.

.....34. Three fourths π radians expressed in degrees equals 120°.

.....35. The sine curve has the same shape as the cosine curve.

.....36. Between 0° and 3600° there are 20 angles with the sine positive.

.....37. When the angle is obtuse, it's functions are those of an angle of the second quadrant.

.....38. The nearer an acute angle is to 0°, the greater the positive value of the tangent.

.....39. If the sec x is 7, the csc is one-seventh.

.....40. When angle is acute, only it's sine and cosine are positive.

Test on Unit IV

Time, 60 minutes

1. a. Solve for x, having given $2 \cos^2 x + 11 \sin x - 7 = 0$. Ans...........
   b. Solve for x, having given $2 \sin^2 x = 3 \cos x$. Ans...........
2. When $\text{Cos } A = 60/61$, $\text{Sin } A =$ .... $\text{Tan } A =$ .... $\text{Cot } A =$ ....
   $\text{Sec } A =$ .... $\text{Csc } A =$ ....
3. When $\text{Tan } 15° = 2 - \sqrt{3}$, $\text{Sin } 15° =$ .... $\text{Cos } 15° =$ ....
   $\text{Cot } 15° =$ .... $\text{Sec } 15° =$ .... $\text{Csc } 15° =$ ....
4. Find all values of X less than 360°.
   a. $\text{Cos } 2X = 5 \text{ Sin }^2X - \text{Cos }^2X$ Ans. X = ...
   b. $\text{Tan } 2X + 2 \text{ Sin } X = 0$ Ans. X = ...
5. If $\text{Tan } 2X = 12/5$, $\text{Sin } \frac{1}{2} X =$ .... $\text{Tan } \frac{1}{2} X =$ ....
6. Prove a. $\text{Sin } 3X = 3 \text{ Sin } X - 4 \text{ Sin }^3X$
   b. Find Sin 5X in terms of Sin X.
7. Given A and B acute angles, with $\text{Sin } A = 12/15$, and $\text{Sin } B = 3/5$
   $\text{Tan } (A + B) =$ ....
   $\text{Tan } (A - B) =$ ....
8. Prove. $\text{Cos}^2 (x-y) - \text{Sin}^2 (x+y) = \text{Cos } 2x \text{ Cos } 2y$
9. Prove. $\text{Sin } 60° + \text{Sin } 30° = 2 \text{ Sin } 45° \text{ Cos } 15°$
10. Prove. $\text{Cos } (45° + x) + \text{Cos } (45° - x) = \sqrt{2 \cdot \text{Cos } x}$

Test on Unit V

Directions: No tables are to be used in this test. The first 20 questions are to be graded 2 points for each question, the last five are to be graded 12 points for each question. Time 40 minutes.

1. Logarithms are ....
2. In the common or decimal system of logarithms the base is ....
3. There are ...... parts to a logarithm. They are ....
4. The..........of a logarithm is decimal in value.
5. In multiplying numbers we ....their logarithms.
6. In finding the root of a number we..............the logarithm of the number by the..............of the...............
7. The logarithm of 1 is ....
8. The logarithm of every number greater than 1 is ....zero.
9. The logarithms of all numbers between 0 and 1 is...........zero.

10. The..........is found by inspection, the..........from a table.

11. The characteristic of a logarithm of a number greater than 1 is.... and is one..................than the number of.............. to the..................of the..................

12. If a number has 4 digits to the left of the decimal point, its characteristic is ....

13. The characteristic of a logarithm of any number between 1 and zero is..............and is one..............than the number of......................between the decimal point and the first significant figure.

14. The number whose logarithm has a characteristic of 4 has........ places to the...................of the decimal point.

15. The characteristic of the logarithm of .0023 is ....

16. If the mantissa of the logarithm of 367 is .5647, then the logarithm of 367 is.................., the log. of 3.67 is ...., the log of .00367 is............

17. The cologarithm of a number is zero..........the............of the number.

18. The cologarithm of x is the logarithm of the fraction (1/x)

19. The logarithm of 81 to the base 3 is ....

20. The logarithm of 128 to the base 2 is ....

21. Given log 3 = .47712, and log 7 = .84510, then log 210 = ...; log $\sqrt{2.1}$ = ...; log $(7/3)^3$ = ...; log $\sqrt{6.3}$ = ...

22. Given the log of 2 = .30103, and the log of 3 = .47712, then log of 180 = ..., log 1/6 = ..., log Sin 30° = ..., log Tan 30° = ..., log Cos 30° = ..., log Sin 45° = ...

23. Given log = .30103, then log 40 = ..., log 50 = ..., log 8 = ..., log 2/5 = ..., log 5/4 = ..., log $\sqrt{20}$ = ...

24. Given log 7 = .84519, log 3 = .47712

then (a) $\sqrt[3]{\dfrac{63^2 \times 81^4 \times \sqrt{49}}{\sqrt{90}}}$ has a logarithm of ..(8 points)

(b) $\sqrt{70} \times \sqrt[5]{7000} \times \sqrt[3]{700}$ has a logarithm of..(4 points)

25. Given log 2 = .30103, log 3 = .47712, log 5 = .69897

(a) then the log of $\sqrt[5]{\dfrac{50^2 \times \sqrt[3]{6000} \times \sqrt{500}}{25 \times 150}}$ = ...(8 points)

(b) then the log of $\sqrt{20} \times \sqrt[3]{1200} \times \sqrt{540}$ = ... (4 points)